C8.2: Stochastic analysis and PDEs
Problem sheet 3

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

Section 1 (Compulsory)

1. Let Y be a Poisson process with parameter λ and define
$$X_n(t) = \frac{1}{n}\big(Y(n^2 t) - \lambda n^2 t\big).$$
Find A_n, the infinitesimal generator for X_n, and identify the limit of the sequence $(A_n)_n$ as $n \to \infty$ and the corresponding stochastic process.

2. In proving that sequences of Markov chains converge to diffusions, we have to verify three conditions on the jumps of the chain. Let us write ΔX^h for the increment of the hth chain over a single jump (in the discrete case in which the time between jumps is h) or over an infinitesimal time interval of length h (in the continuous case). Our conditions amount to checking that $\mathbb{E}[\Delta X^h]/h$ and $\mathbb{E}[(\Delta X^h)^2]/h$ both converge as $h \to 0$ and that $\mathbb{P}[|\Delta X^h| > \epsilon]/h \to 0$ as $h \to 0$. Prove that this last condition is implied by the (often more convenient) condition $\mathbb{E}[(\Delta X^h)^4]/h \to 0$.

3. Let $Af(x) = \frac{1}{2}x(1-x)f''(x) + (a-bx)f'(x)$, for f twice continuously differentiable on $[0,1]$, where a and b are positive constants. This is the generator of the *Wright-Fisher diffusion with mutation.* Use duality to show uniqueness of the solution to the corresponding martingale problem and to show that, if X is a solution, $X(t)$ converges in distribution as $t \to \infty$ with a limiting distribution which does not depend on $X(0)$. *[Hint: Try a duality function of the form $F(x,n) = x^n$ where n is a non-negative integer and the extended duality result in the notes.]*

4. Let $E_1 = E_2 = [0,\infty)$. We take A_1 to be the generator of Brownian motion on $[0,\infty)$ with absorbing boundary condition, that is $A_1 f = \frac{1}{2}f''$ with
$$\mathcal{D}(A_1) = \{f \in \bar{\mathcal{C}}^2 : f''(0) = 0\}.$$
(Here $\bar{\mathcal{C}}^2$ indicates bounded and twice continuously differentiable.) We take A_2 to be the generator of reflecting Brownian motion on $[0,\infty)$, so
$$\mathcal{D}(A_1) = \{f \in \bar{\mathcal{C}}^2 : f'(0) = 0\}.$$
   (a) Let $g \in \bar{\mathcal{C}}^2(-\infty,\infty)$ satsify $g(z) = -g(-z)$. Show that the martingale problems for A_1 and A_2 are dual with respect to $F(x,y) = g(x+y) + g(x-y)$.

(b) Use the result above to show that

$$\mathbb{P}[X(t) > y | X(0) = x] = \mathbb{P}[Y(t) < x | Y(0) = y],$$

where X is absorbing Brownian motion and Y is reflecting Brownian motion.

5. Consider the Wright-Fisher diffusion with mutation, whose generator can be written as

$$Af(x) = \frac{1}{2}x(1-x)f''(x) + \big(\nu_2 - (\nu_1 + \nu_2)x\big)f'(x)$$

on suitable test functions. Find conditions on ν_2 for the boundary at 0 to be *regular*.

6. The *Bessel process* with parameter $\alpha \geq 0$ is the one-dimensional diffusion on $[0, \infty)$ with generator

$$Af(x) = \frac{1}{2}f''(x) + \frac{\alpha - 1}{2x}f'(x).$$

(When α is an integer, it is the radial part of a Brownian motion in $\mathbb{R}^\alpha$.) Find expressions for the speed and scale and hence determine the nature of the boundary at 0 for each parameter value α.

7. Suppose that we are modelling a population in which each individual carries a particular gene in one of two forms, which we label a and A. We assume that each individual has exactly one parent, and offspring inherit the genetic type of their parent. We consider two models:

(a) In the *neutral Wright-Fisher model* a population of N genes evolves in discrete generations. Generation $(t+1)$ is formed from generation t by choosing N genes uniformly at random with replacement. i.e. each gene in generation $(t+1)$ chooses its parent independently at random from those present in generation t. Let us write $X_t^{(N)}$ for the proportion of type a genes in the population at time t under this model.

(b) In the *neutral Moran model*, generations overlap. At exponential rate $\binom{N}{2}$ a pair of genes is sampled uniformly at random from the population. One of the pair is selected at random to die and the other splits in two. Let us write $Y_t^{(N)}$ for the proportion of type a genes in the population at time t under this model.

Show that the processes $X_{Nt}^{(N)}$ and $Y_t^{(N)}$ both converge as $N \to \infty$ and identify the limiting diffusion.

Section 2 (Extra practice questions, not for hand-in)

A. Suppose that X is a *Feller branching process*, that is $X \geq 0$ solves the SDE

$$dX_t = aX_t dt + \sqrt{\gamma X_t} dB_t,$$

for suitable constants $a \in (-\infty, \infty)$ and $\gamma > 0$. By considering the duality function $F(X, Y) = \exp(-XY)$, show that X has a *deterministic* dual and use it to find $\mathbb{P}[X_t \neq 0]$.

B. Consider the *Wright-Fisher diffusion with selection*, which has generator

$$Af = \frac{1}{2}x(1-x)f''(x) + sx(1-x)f'(x),$$

for suitable $\mathcal{C}^2$ functions f on $[0, 1]$ where s is a constant (called the *selection coefficient*). Use duality to check that the martingale problem has a unique solution.

C. The Ornstein-Uhlenbeck process on $\mathbb{R}$ is the one-dimensional diffusion with generator

$$Af(x) = \frac{1}{2}f''(x) - xf'(x).$$

Prove that both ∞ and $-\infty$ are natural boundaries.

D. Our diffusion approximation to the neutral Wright-Fisher model with no mutation suggests that for a large population, the time, $\tau(p)$, to absorption in one of the states $\{0,1\}$ if the initial frequency of a-alleles is p, should be approimately $t(p)$ given by

$$t(p) = -2N\left(p\log p + (1-p)\log(1-p)\right). \tag{1}$$

By conditioning on the behaviour of allele-frequencies over a single generation, and using a Taylor exansion, find an ordinary differential equation that approximates the behaviour of $\tau(p)$ (as a function of p) and verify that it leads to the same approximation.

E. A Galton Watson branching process is a discrete time Markov chain, $\{Z_n\}_{n\geq 1}$, which is often used to model the growth of a population. The evolution is simple. Each individual leaves behind a random number of offspring in the following generation, according to some distribution, independently of all other individuals. Suppose that the mean number of offspring of each individual is a and the variance is σ^2 and write Z_0 for the initial population size.

   (a) What is the expected population size after N generations?

   (b) If we are modelling a very large population, whose size at time zero is NX_0 for some large N and $a \approx 1 + \frac{\mu}{N}$, then find a diffusion approximation for the population size at time Nt in units of size N.