C8.2: Stochastic analysis and PDEs
Solutions to Problem sheet 4

The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

Section 1 (Compulsory)

1. *Let r satisfy the stochastic differential equation*

$$dr_t = -\beta r_t dt + \sigma\sqrt{r_t}dW_t,$$

where $\{W_t\}_{t\geq 0}$ is standard $\mathbb{P}$-Brownian motion and $\beta, \sigma, r_0 > 0$.

Suppose that $\{u(t)\}_{t\geq 0}$ satisfies the ordinary differential equation

$$\frac{du}{dt}(t) = -\beta u(t) - \frac{\sigma^2}{2}u(t)^2, \quad u(0) = \theta,$$

for some constant $\theta > 0$. Fix $T > 0$. For $0 \leq t \leq T$ find the stochastic differential equation satisfied by

$$\exp\left(-u(T-t)r_t\right).$$

Hence find the moment generating function for r_T. Calculate the mean and variance of r_T and $\mathbb{P}[r_T = 0]$.

It is not hard to check that 0 is an exit boundary for r. Thus $r_t \geq 0$ for all $t \geq 0$.

Applying Ito's formula to $M_t = \exp\left(-u(T-t)r_t\right)$, we have

$$dM_t = (u'(T-t) + u(T-t)\beta + \frac{\sigma^2}{2}u(T-t)^2)r_t M_t dt - \sigma u(T-t)\sqrt{r_t}M_t dW_t\,.$$

Using the fact that u satisfies the ODE we have that $dM_t = -\sigma u(T-t)\sqrt{r_t}M_t dW_t$, so that M is a local martingale. As the solution to the ODE is bounded (since $\frac{du}{dt}$ is negative) and r is positive we have $M \leq 1$. Thus M is a bounded local martingale and hence a martingale. Taking expectations we have

$$\mathbb{E}\left[\exp(-u(0)r_T)\right] = \exp(-u(T)r_0).$$

To find the moment generating function for r_T, it thus suffices to solve the ODE for $u(T)$. As $u(0) = \theta$, we have

$$u(T) = \frac{\theta e^{-\beta T}}{1 + \gamma\theta - \gamma\theta e^{-\beta T}},$$

where $\gamma = \sigma^2/2\beta$. Then

$$\psi_t(\theta) := \mathbb{E}\left[\exp(-\theta r_T)\right] = \mathbb{E}\left[\exp(-u(0)r_T)\right] = \exp\Big(-\frac{r_0\theta e^{-\beta T}}{1 + \gamma\theta - \gamma\theta e^{-\beta T}}\Big).$$

By differentiating the MGF $\psi_t(\theta)$ we have $\mathbb{E}[r_T] = r_0 e^{-\beta T}$, $var(r_T) = r_0\sigma^2 e^{-\beta T}(1-e^{-\beta T})/\beta$. Note that, for every $x \geq 0$,

$$\lim_{\theta\to\infty} e^{-\theta x} = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Therefore, by dominated convergence,

$$\mathbb{P}[r_T = 0] = \lim_{\theta\to\infty} \mathbb{E}[\exp(-\theta r_T)] = \exp\left(-r_0 \frac{2\beta e^{-\beta T}}{\sigma^2\left(1-e^{-\beta T}\right)}\right).$$

2. *Use the Feynman-Kac stochastic representation formula to solve*

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0,$$

subject to the terminal value condition

$$F(T,x) = x^4.$$

Assuming that the Feynman-Kac representation is applicable (which is not immediately clear since x^4 is unbounded), we have $F(t,x) = \mathbb{E}\left[\left. X_T^4 \right| X_t = x\right]$, where $X_t = \sigma W_t$. Using properties of the normal distribution, this expectation is given by $F(t,x) = 3\sigma^4(T-t)^2 + 6\sigma^2(T-t)x^2 + x^4$. One can now readily see that this choice for F is indeed a solution.

3. *We can use the Feynman-Kac representation to find the partial differential equation solved by the transition densities of solutions to stochastic differential equations.*

Suppose that

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t. \tag{1}$$

For any set B let

$$p_B(t,x;T) \triangleq \mathbb{P}\left[\left. X_T \in B \right| X_t = x\right] = \mathbb{E}\left[\left. \mathbf{1}_B(X_T) \right| X_t = x\right].$$

Use the Feynman-Kac representation (assuming integrability conditions are satisfied) to write down an equation for

$$\frac{\partial p_B}{\partial t}(t,x;T)$$

Deduce that the transition density of the solution $\{X_s\}_{s\geq 0}$ to the stochastic differential equation (1) solves

$$\begin{aligned} \frac{\partial p}{\partial t}(t,x;T,y) + Ap(t,x;T,y) &= 0 \qquad (2) \\ p(t,x;T,y) &\to \delta_y(x) \quad \textit{as } t \to T, \end{aligned}$$

where A is the generator. Equation (2) is known as the Kolmogorov backward equation *(it operates on the 'backward in time' variables (t,x)).*

By the Feynman-Kac representation (subject to the integrability condition)

$$\begin{aligned} \frac{\partial p_B}{\partial t}(t,x;T) + Ap_B(t,x;T) &= 0 \qquad (3) \\ p_B(T,x;T) &= \mathbf{1}_B(x), \end{aligned}$$

where

$$Af(t,x) = \mu(t,x)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x).$$

Writing $|B|$ for the Lebesgue measure of the set B, the transition density of the process $\{X_s\}_{s\geq 0}$ is given by

$$p(t,x;T,y) \triangleq \lim_{B\to y} \frac{1}{|B|}\mathbb{P}\left[X_T \in B \middle| X_t = x\right].$$

(We are assuming existence of the density). Since the equation (3) is linear, we have proved that the transition density of the solution $\{X_s\}_{s\geq 0}$ to the stochastic differential equation (1) solves (2) as required.

We can also obtain an equation acting on the forward *variables* (T,y)*. In the above notation,*

$$\frac{\partial p}{\partial T}(t,x;T,y) = A^* p(t,x;T,y) \tag{4}$$

where

$$A^* f(T,y) = -\frac{\partial}{\partial y}\left(\mu(T,y)f(T,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,Y)f(T,y)\right).$$

Heuristic explanation: *By the Markov property of the process* $\{X_t\}_{t\geq 0}$*, for any* $T > r > t$

$$p(t,x;T,y) = \int p(t,x;r,z)p(r,z;T,y)dz.$$

Differentiating with respect to r *and using (2),*

$$\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial r}p(t,x;r,z)p(r,z;T,y) - p(t,x;r,z)Ap(r,z;T,y)\right\}dz = 0.$$

Now integrate the second term by parts to obtain

$$\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial r}p(t,x;r,z) - A^* p(t,x;r,z)\right\}p(r,z;T,y)dz = 0.$$

This holds for all $T > r$*, which, if* $p(r,z;T,y)$ *provides a sufficiently rich class of functions as we vary* T*, implies the result.*

Equation (4) is the Kolmogorov forward equation *of the process* $\{X_s\}_{s\geq 0}$.

4. *Suppose that* $\{X_t\}_{t\geq 0}$ *solves*

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t,$$

where $\{W_t\}_{t\geq 0}$ *is a* $\mathbb{P}$*-Brownian motion. For* $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *and* $\Phi : \mathbb{R} \to \mathbb{R}$ *given deterministic functions, find the partial differential equation satisfied by the function*

$$F(t,x) \triangleq \mathbb{E}\left[\exp\left(-\int_t^T k(s,X_s)ds\right)\Phi(X_T)\middle| X_t = x\right],$$

for $0 \leq t \leq T$.

Evidently $F(T,x) = \Phi(x)$. By analogy with the proof of the Feynman-Kac representation, it is tempting to examine the dynamics of

$$Z_s = \exp\left(-\int_t^s k(u,X_u)du\right)F(s,X_s).$$

Notice that if this choice of $\{Z_s\}_{t\leq s\leq T}$ is a martingale we have that

$$Z_t = F(t,x) = \mathbb{E}\left[\, Z_T \,|\, X_t = x\right].$$

Thus the partial differential equation satisfied by $F(t,x)$ is that for which $\{Z_t\}_{0\leq t\leq T}$ is a martingale.

Our strategy now is to find the stochastic differential equation satisfied by $\{Z_s\}_{t\leq s\leq T}$. We proceed in two stages. Remember that t is now fixed and we vary s. First notice that

$$d\left(\exp\left(-\int_t^s k(u,X_u)du\right)\right) = -k(s,X_s)\exp\left(-\int_t^s k(u,X_u)du\right)ds$$

and by Itô's formula

$$\begin{aligned} dF(s,X_s) &= \frac{\partial F}{\partial s}(s,X_s)ds + \frac{\partial F}{\partial x}(s,X_s)dX_s + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s,X_s)\sigma^2(s,X_s)ds \\ &= \left\{\frac{\partial F}{\partial s}(s,X_s) + \mu(s,X_s)\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2}(s,X_s)\right\}ds \\ &\qquad + \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s. \end{aligned}$$

Hence

$$\begin{aligned} dZ_s = \exp&\left(-\int_t^s k(u,X_u)du\right)\times \\ &\left\{\left\{-k(s,X_s)F(s,X_s) + \frac{\partial F}{\partial s}(s,X_s) + \mu(s,X_s)\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}\sigma^2(s,X_S)\frac{\partial^2 F}{\partial x^2}\right\}ds \right. \\ &\qquad \left. + \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s\right\}. \end{aligned}$$

We can now read off the solution: $\{Z_s\}_{t\leq s\leq T}$ will be a martingale if F satisfies

$$\frac{\partial F}{\partial s}(s,x) + \mu(s,x)\frac{\partial F}{\partial x}(s,x) + \frac{1}{2}\sigma^2(s,x)\frac{\partial^2 F}{\partial x^2}(s,x) - k(s,x)F(s,x) = 0.$$

5. *Let B be a Brownian motion in $\mathbb{R}$ and consider $A_t = \int_0^t I_{\{B_u>0\}}du$, the amount of time that Brownian motion spends in the positive half line up to time t. Let $F(t,x) = \mathbb{E}(\exp(-\theta A_t)|B_0 = x)$, the Laplace transform of A_t given that the Brownian motion starts from x. By setting $r(t,B_t) = -\theta I_{\{B_t>0\}}$ (in the Feynman-Kac formula in the notes) and $\Psi = 1$ and using a time reversed version of the Feynman-Kac formula, show the PDE satisfied by F, is*

$$\frac{\partial F}{\partial t} = \begin{cases} \frac{1}{2}\frac{\partial^2 F}{\partial x^2} - \theta F & x>0, t>0 \\ \frac{1}{2}\frac{\partial^2 F}{\partial x^2} & x\leq 0, t>0. \end{cases}$$

specifying the initial conditions and, carefully, the continuity conditions at 0. By taking Laplace transforms, $\hat{F}(\lambda,x) = \int_0^\infty \exp(-\lambda t)F(t,x)dt$ and solving the resulting ODE, show that

$$\hat{F}(\lambda,0) = \frac{1}{\sqrt{\lambda}\sqrt{\lambda+\theta}}. \tag{5}$$

From this we can derive Levy's arcsine law,

$$P(A_t \le s | X_0 = 0) = \int_0^s \frac{1}{\pi\sqrt{u(t-u)}} du = \frac{2}{\pi} \arcsin(\sqrt{\frac{s}{t}}), \quad 0 \le s \le t.$$

To see this compute the Laplace transform of the arcsine law by suitably integrating to show that the transform is as given in (5).

Let

$$A_t = \int_0^t I_{B_u > 0} du,$$

and

$$\begin{aligned} F(t,x) &= \mathbb{E}\left[e^{-\theta A_t} | B_0 = x\right] \\ &= \mathbb{E}\left[e^{-\theta \int_0^t I_{B_u>0} du} | B_0 = x\right]. \end{aligned}$$

Now suppose f satisfies the following PDE for some function r,

$$\frac{\partial f}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) + r(x) f(t,x), \qquad (t,x) \in (0,\infty) \times \mathbb{R}$$

with initial condition

$$f(0,x) = \Psi(x).$$

This is a heat equation with dissipation, and the Feynman-Kac formula gives that

$$f(t,x) = \mathbb{E}\Big[\exp\Big(-\int_0^t r(B_u) du\Big)\Psi(B_t) | B_0 = x.\Big]$$

Hence for our function F given by

$$F(t,x) = \mathbb{E}\Big[\exp\Big(-\theta\int_0^t I_{B_u>0} du\Big)\Psi(B_t) | B_0 = x\Big]$$

the PDE is, for $t > 0$,

$$\frac{\partial F}{\partial t} = \begin{cases} \frac{1}{2}\frac{\partial^2 F}{\partial x^2} - \theta F, & \text{for } x > 0 \\ \frac{1}{2}\frac{\partial^2 F}{\partial x^2}, & \text{for } x \le 0 \end{cases}$$

with $F(0,x) = 1$ and continuity at 0, in that $F(t,0+) = F(t,0-)$ and $\frac{\partial F}{\partial x}(t,0+) = \frac{\partial F}{\partial x}(t,0-)$. (Strictly speaking, in the lectures we only considered $F \in C^2$ and we had to know a priori that a solution to the given PDE exists; here $\frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial^2 F}{\partial x^2}$ will have a discontinuity at $x = 0$ and it is not immediately clear that a solution to the PDE exists. We can rest assured that these technical points do not cause any fundamental problems.) Now consider the Laplace transform

$$\hat{F}(\lambda, x) = \int_0^\infty e^{-\lambda t} F(t,x) dt.$$

Then, denoting $\hat{F}''(\lambda,x)$ the second derivative in x of $\hat{F}(\lambda,x)$,

$$\lambda\hat{F}(\lambda,x) - F(0,x) = \begin{cases} \frac{1}{2}\hat{F}''(\lambda,x) - \theta\hat{F}(\lambda,x) & \text{for } x > 0 \\ \frac{1}{2}\hat{F}''(\lambda,x), & \text{for } x \le 0 \end{cases}$$

$$\text{i.e. } \hat{F}''(\lambda, x) - 2\lambda\hat{F}(\lambda, x) - 2\theta\hat{F}(\lambda, x) + 2 = 0, \qquad x > 0$$
$$\text{and } \hat{F}''(\lambda, x) - 2\lambda\hat{F}(\lambda, x) + 2 = 0, \qquad x \le 0.$$

Solving these ODEs, for $x \le 0$,

$$\hat{F}(\lambda, x) = A_1 e^{\sqrt{2\lambda}x} + B_1 e^{-\sqrt{2\lambda}x} + \frac{1}{\lambda},$$

and for $x > 0$,

$$\hat{F}(\lambda, x) = A_2 e^{\sqrt{2(\lambda+\theta)}x} + B_2 e^{-\sqrt{2(\lambda+\theta)}x} + \frac{1}{\lambda+\theta}.$$

To determine A_1, B_1, A_2, B_2, observe that

$$F(t, x) \to \begin{cases} e^{-\theta t} & \text{as } x \to \infty \\ 1, & \text{as } x \to -\infty, \end{cases}$$

hence

$$\hat{F}(\lambda, x) \to \begin{cases} \frac{1}{\lambda+\theta} & \text{as } x \to \infty \\ \frac{1}{\lambda}, & \text{as } x \to -\infty. \end{cases}$$

Also by the continuity condition $F(t, 0+) = F(t, 0-)$ and $\frac{\partial F}{\partial x}(t, 0+) = \frac{\partial F}{\partial x}(t, 0-)$ we get

$$\hat{F}(\lambda, 0+) = \hat{F}(\lambda, 0-)$$

and

$$\hat{F}'(\lambda, 0-) = \hat{F}'(\lambda, 0+).$$

Therefore

$$x \to -\infty \quad \Rightarrow \quad B_1 = 0,$$
$$x \to \infty \quad \Rightarrow \quad A_2 = 0 \quad \text{and}$$
$$A_1 + \frac{1}{\lambda} = B_2 + \frac{1}{\lambda+\theta} \Rightarrow A_1 = B_2 + \frac{1}{\lambda+\theta} - \frac{1}{\lambda}$$
$$\sqrt{2\lambda}A_1 = -\sqrt{2(\lambda+\theta)}B_2.$$

Finally

$$\sqrt{2\lambda}\left(B_2 - \frac{\theta}{\lambda(\lambda+\theta)}\right) = -\sqrt{2(\lambda+\theta)}B_2$$
$$\left(\sqrt{\lambda} + \sqrt{(\lambda+\theta)}\right) B_2 = \frac{\theta}{\sqrt{\lambda}(\lambda+\theta)}$$
$$B_2 = \frac{\theta}{\sqrt{\lambda}(\lambda+\theta)(\sqrt{\lambda} + \sqrt{(\lambda+\theta)})}$$

$$\begin{aligned}
\hat{F}(\lambda,0) &= \frac{\theta}{\sqrt{\lambda}(\lambda+\theta)\left(\sqrt{\lambda}+\sqrt{(\lambda+\theta)}\right)} + \frac{1}{\lambda+\theta} \\
&= \frac{1}{\lambda+\theta}\left[\frac{\theta+\lambda+\sqrt{\lambda(\lambda+\theta)}}{\lambda+\sqrt{\lambda(\lambda+\theta)}}\right] \\
&= \frac{1}{\lambda+\theta}\left[\frac{\theta(\lambda-\sqrt{\lambda(\lambda+\theta)})+\lambda^2-(\lambda^2+\lambda\theta)}{-\lambda\theta}\right] \\
&= \frac{1}{\sqrt{\lambda(\lambda+\theta)}},
\end{aligned}$$

as required.

To show that this is the Laplace transform for the arcsin law, consider the random variable Y_t with

$$\mathbb{P}(Y_t \in ds) = \frac{ds}{\pi\sqrt{s(t-s)}}, \quad 0 < s < t$$

$$\text{so } \mathbb{E}e^{-\theta Y_t} = \int_0^t e^{-\theta s}\frac{ds}{\pi\sqrt{s(t-s)}} = F(t,0)$$

$$\begin{aligned}
\hat{F}(\lambda,0) &= \int_0^\infty \int_0^t e^{-\lambda t-\theta s}\frac{ds}{\pi\sqrt{s(t-s)}}dsdt \\
&= \int_{s=0}^\infty \int_s^\infty e^{-\lambda t-\theta s}\frac{ds}{\pi\sqrt{s(t-s)}}dtds \\
(u=t-s) \quad &= \int_0^\infty \int_0^\infty e^{-\lambda u-(\theta+\lambda)s}\frac{1}{\pi\sqrt{s}\sqrt{u}}duds \\
&= \frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda u}}{\sqrt{u}}du \int_0^\infty \frac{e^{-(\theta+\lambda)s}}{\sqrt{s}}ds
\end{aligned}$$

Using the substitution $a = \sqrt{2u}$, then $u = \frac{a^2}{2}$ and $du = ada$, we have

$$\int_0^\infty \frac{e^{-\lambda u}}{\sqrt{\pi u}}du = \frac{1}{\sqrt{\lambda}}.$$

In the same way we have

$$\int_0^\infty \frac{e^{-(\theta+\lambda)s}}{\sqrt{\pi s}}ds = \frac{1}{\sqrt{\theta+\lambda}},$$

and hence

$$\hat{F}(\lambda,0) = \frac{1}{\sqrt{\lambda}\sqrt{\theta+\lambda}}.$$

Thus as the Laplace transform of A is given by this expression we must have

$$\mathbb{P}(A_t \in ds) = \frac{ds}{\pi\sqrt{s}\sqrt{t-s}}.$$

Section 2 (Extra practice questions, not for hand-in)

A. *Consider a three dimensional Brownian motion started at the origin and stopped at the first time it exists the unit sphere. Fix* $0 < r < 1$. *In which of the annuli*

$$A[a] = \{x \in \mathbb{R}^3 \,:\, a - r \le |x| \le a\} \text{ for } a \in [r,1]$$

is the expected occupation time maximal?

Denoting by U the unit ball and $T = \inf\{t > 0 : B_t \in \partial U\}$, recall that

$$\mathbb{E}_0\Big\{\int_0^T \mathbf{1}_{A[a]}(B_s)\,ds\Big\} = \int_{A[a]} G_U(0,y)\,dy\ ,$$

where $G_U(x,y) = G(x,y) - \int_{\partial U} G(z,y)\mu(x,dz)$ is the Green's function of U, $\mu(x,dz) = \mathbb{P}[B_T \in dz] = \frac{1-|x|^2}{|x-z|^3}\pi(dz)$ is the Harmonic measure and π is the uniform measure on ∂U, and $G(x,y)$ is the potential kernel given by

$$G(x,y) = C|x-y|^{-1}$$

with constant $C = \frac{\Gamma(\frac{3}{2}-1)}{2\pi^{3/2}}$. Recall further that

$$\int_{\partial U} \frac{1-|x|^2}{|x-z|^3} G(z,y)\pi(dz) = C\frac{|y|}{|x|y|^2 - y|}\ .$$

It follows that

$$\int_{A[a]} G_U(0,y)\,dy = \int_{A[a]} (C|y|^{-1} - C)\,dy\ .$$

We can do the calculation in polar coordinates:

$$\begin{aligned}
\int_{A[a]} (C|y|^{-1} - C)\,dy &= C4\pi\int_{a-r}^{a}\Big(\frac{1}{r} - 1\Big)r^2\,dr \\
&= C4\pi\Big(\frac{a^2}{2} - \frac{a^3}{3} - \frac{(a-r)^2}{2} + \frac{(a-r)^3}{3}\Big) \\
&= C4\pi\Big(r(a-a^2) + r^2(a-1/2) - r^3/3\Big)\ .
\end{aligned}$$

We note that this is maximal if $r(1-2a) + r^2 = 0$ i.e. exactly if $a = (1+r)/2$, if the annulus is in the middle. Note that the expected occupation time in the inner ball $a = r$ and the outer annulus $a = 1$ are the same.

B. *Suppose that* $v(t,x)$ *solves*

$$\frac{\partial v}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(t,x) - rv(t,x) = 0, \quad 0 \le t \le T.$$

Show that for any constant θ,

$$v_\theta(t,x) \triangleq \frac{x}{\theta} v\left(t, \frac{\theta^2}{x}\right)$$

is another solution.

Probabilistically, the point is that for a geometric Brownian motion $\{X_t\}_{0\le t\le T}$, dependence of X_T on X_t is only as a multiplier.

C. *Suppose that for* $0 \leq s \leq T$,

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \quad X_t = x,$$

where $\{W_s\}_{t \leq s \leq T}$ *is a* $\mathbb{P}$*-Brownian motion, and let* $k, \Phi : \mathbb{R} \to \mathbb{R}$ *be given deterministic functions. Find the partial differential equation satisfied by*

$$F(t, x) = \mathbb{E}\left[\Phi(X_T) \middle| X_t = x\right] + \int_t^T \mathbb{E}\left[k(X_s) \middle| X_t = x\right] ds.$$

Using the same reasoning as question 4 we apply Itô's formula to $F(s, X_s) + \int_t^s k(X_u)du$ and integrate with respect to s over $[t, T]$ to see that

$$\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} + k = 0,$$

and $F(T, x) = \Phi(x)$.

D. *In the* Vasicek model, *the interest rate* $\{r_t\}_{t \geq 0}$ *is assumed to be a solution of the stochastic differential equation*

$$dr_t = (b - ar_t)dt + \sigma dW_t,$$

where, as usual, $\{W_t\}_{t \geq 0}$ *is standard* $\mathbb{P}$*-Brownian motion.*

Find the Kolmogorov backward and forward differential equations satisfied by the probability density function of the process. What is the distribution of r_t *as* $t \to \infty$*?*

$$\frac{\partial p(t, T; x, y)}{\partial t} = -\frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} - (b - ax)\frac{\partial p}{\partial x}.$$

$$\frac{\partial p(t, T; x, y)}{\partial T} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial y^2} - \frac{\partial}{\partial y}\left((b - ay)p\right).$$

Consider $u_t = e^{at} r_t$.

$$du_t = be^{at}dt + \sigma e^{at}dW_t.$$

Integrating and substituting back gives

$$r_t = e^{-at}r_0 + e^{-at}\int_0^t be^{as}ds + \int_0^t \sigma e^{-a(t-s)}dW_s.$$

Thus r_t is normally distributed with mean $e^{-at}r_0 + \frac{b}{a}(1 - e^{-at})$ and variance $\frac{\sigma^2}{2a}(1 - e^{-2at})$. As $t \to \infty$, r_t tends to a normally distributed random variable with mean b/a and variance $\sigma^2/2a$.

E. *The process usually known as* Geometric Brownian motion *solves the s.d.e.*

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Find the forward and backward Kolmogorov equations for geometric Brownian motion and show that the transition density for the process is the lognormal density given by

$$p(t, x; T, y) = \frac{1}{\sigma y \sqrt{2\pi(T - t)}} \exp\left(-\frac{\left(\log(y/x) - \left(\mu - \frac{1}{2}\sigma^2\right)(T - t)\right)^2}{2\sigma^2(T - t)}\right).$$

Substituting in our formula for the forward equation we obtain

$$\frac{\partial p}{\partial T}(t,x;T,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(y^2 p(t,x;T,y)\right) - \mu\frac{\partial}{\partial y}\left(yp(t,x;T,y)\right),$$

and the backward equation is

$$\frac{\partial p}{\partial t}(t,x;T,y) = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2}(t,x;T,y) - \mu x\frac{\partial p}{\partial x}(t,x;T,y).$$

It is enough to check that the lognormal density solves one of the Kolmogorov equations.