C8.2 Stochastic Analysis and PDEs
2016 Q2

2. Let E be a complete separable metric space and write $C(E)$ for the bounded, continuous, real-valued functions on E with the supremum norm.

(a) What does it mean to say that an operator A on $C(E)$ is a *Markov generator*?

Definition 2.18 for Markov pregenerator, Definition 2.30 for Markov generator

(b) Let A be a Markov generator and let $x \in E$. What does it mean to say that a probability measure $\mathbb{P}$ solves the martingale problem for A with initial point x?

Definition 3.2: A probability measure $\mathbb{P}$ on $D[0, \infty)$ solves the martingale problem for A with initial point x if

1. $\mathbb{P}[\{X \in D : X_0 = x\}] = 1$, and
2. for all $f \in \mathcal{D}(A)$

$$f(X_t) - \int_0^t Af(X_s)\,\mathrm{d}s$$

is a (local) martingale relative to $\mathbb{P}$ and the canonical filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by the projections.

(c) Suppose that A is a Markov generator and let $\mathbb{P}^x$ be the law of the unique Feller process corresponding to A started from x. Show that $\mathbb{P}^x$ solves the martingale problem for A with initial point x.

First half of the proof of Theorem 3.3

(d) Consider the $[-1, 1]$-valued diffusion $\{X_t\}_{t \geq 0}$ with infinitesimal generator

$$Af(x) = \frac{1}{2}(1 - x^2) f''(x),$$

when restricted to an appropriate subset of the twice continuously differentiable functions on $[-1, 1]$. By considering the martingale problem with suitable functions,

(i) show that $X_\infty = \lim_{t \to \infty} X_t$ exists and find its expectation;

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Choose $f(x) = x$. Then $Af(x) = 0$, hence X_t is a bounded local martingale, and thus $\lim_{t\to\infty} X_t = X_\infty$ exists a.s. and

$$\mathbb{E}^x[X_\infty] = \mathbb{E}^x[X_0] = x\ .$$

(d) (ii) show that $\mathbb{P}[X_\infty \in \{-1, 1\}] = 1$ and for $x \in (-1, 1)$ find $\mathbb{P}[X_\infty = 1 | X_0 = x]$.

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Finally,

$$\begin{aligned} x &= \mathbb{E}^x[X_\infty] \\ &= \mathbb{P}^x[X_\infty = 1] - \mathbb{P}^x[X_\infty = -1] \\ &= \mathbb{P}^x[X_\infty = 1] - (1 - \mathbb{P}^x[X_\infty = 1])\ . \end{aligned}$$

Hence $\mathbb{P}^x[X_\infty = 1] = \frac{1+x}{2}$.