C8.2 Stochastic Analysis and PDEs
2017 Q1

1.(a)(i) Define what it means for $\{P_t; t \geq 0\}$, a family of bounded operators on a Banach space C, to be a *strongly continuous contraction semi-group*.

Definition 2.9

1.(a)(i) Define what it means for $\{P_t; t \geq 0\}$, a family of bounded operators on a Banach space C, to be a *strongly continuous contraction semi-group*.

Definition 2.9

(ii) Let $A_t = \frac{1}{t}(P_t - I)$ for $t > 0$. Use this to define A, the infinitesimal generator of the semi-group, and its domain $\mathcal{D}(A)$.

Proposition 2.14: $\mathcal{D}(A) = \{z \in C \,:\, \lim_{t\to\infty} A_t z \text{ exits}\}$ and $Az = \lim_{t\to 0} A_t z$.

1.(a)(i) Define what it means for $\{P_t; t \geq 0\}$, a family of bounded operators on a Banach space C, to be a *strongly continuous contraction semi-group*.

Definition 2.9

(ii) Let $A_t = \frac{1}{t}(P_t - I)$ for $t > 0$. Use this to define A, the infinitesimal generator of the semi-group, and its domain $\mathcal{D}(A)$.

Proposition 2.14: $\mathcal{D}(A) = \{z \in C : \lim_{t\to\infty} A_t z \text{ exits}\}$ and $Az = \lim_{t\to 0} A_t z$.

(iii) Show that, for all $f \in \mathcal{D}(A)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t f = AP_t f.$$

Second part of the proof of Proposition 2.14.

(b) Let $\{R_\lambda; \lambda > 0\}$ be the resolvent for the semi-group $\{P_t; t \geq 0\}$, where $R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f \, \mathrm{d}t$ for $f \in C$.

(i) Show that $\lambda R_\lambda = \mathbb{E} P_\tau$ where τ is an exponentially distributed random variable with parameter λ.

(b) Let $\{R_\lambda; \lambda > 0\}$ be the resolvent for the semi-group $\{P_t; t \geq 0\}$, where $R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f \, \mathrm{d}t$ for $f \in C$.

(i) Show that $\lambda R_\lambda = \mathbb{E} P_\tau$ where τ is an exponentially distributed random variable with parameter λ.

A: Using the density of the law of τ

$$\mathbb{E}[P_\tau f] = \int_0^\infty \lambda e^{-\lambda t} P_t f \, \mathrm{d}t = \lambda R_\lambda f$$

(b) Let $\{R_\lambda; \lambda > 0\}$ be the resolvent for the semi-group $\{P_t; t \geq 0\}$, where $R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f \,\mathrm{d}t$ for $f \in C$.

(i) Show that $\lambda R_\lambda = \mathbb{E} P_\tau$ where τ is an exponentially distributed random variable with parameter λ.

A: Using the density of the law of τ

$$\mathbb{E}[P_\tau f] = \int_0^\infty \lambda e^{-\lambda t} P_t f \,\mathrm{d}t = \lambda R_\lambda f$$

(ii) Show that λR_λ is a contraction on C.

(b) Let $\{R_\lambda; \lambda > 0\}$ be the resolvent for the semi-group $\{P_t; t \geq 0\}$, where $R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f \, \mathrm{d}t$ for $f \in C$.

(i) Show that $\lambda R_\lambda = \mathbb{E} P_\tau$ where τ is an exponentially distributed random variable with parameter λ.

A: Using the density of the law of τ

$$\mathbb{E}[P_\tau f] = \int_0^\infty \lambda e^{-\lambda t} P_t f \, \mathrm{d}t = \lambda R_\lambda f$$

(ii) Show that λR_λ is a contraction on C.

A: Using part (i), for any $f \in C$

$$\begin{aligned} \|\lambda R_\lambda f\| = \|\mathbb{E}[P_\tau f]\| &\leq \mathbb{E}[\|P_\tau f\|] \\ &\leq \|f\| \quad (P_t \text{ is contraction } \forall t \geq 0). \end{aligned}$$

(b)(iii) Show that $\lambda R_\lambda \to I$ as $\lambda \to \infty$.

(b)(iii) Show that $\lambda R_\lambda \to I$ as $\lambda \to \infty$.

A: For every $f \in C$,

$$\lambda R_\lambda f = \int_0^\infty \lambda e^{-\lambda t} P_t f \,\mathrm{d}t = \int_0^\infty e^{-s} P_{s/\lambda} f \,\mathrm{d}s \ .$$

(b)(iii) Show that $\lambda R_\lambda \to I$ as $\lambda \to \infty$.

A: For every $f \in C$,

$$\lambda R_\lambda f = \int_0^\infty \lambda e^{-\lambda t} P_t f \,\mathrm{d}t = \int_0^\infty e^{-s} P_{s/\lambda} f \,\mathrm{d}s \ .$$

For every $s \geq 0$, by strong continuity of P_t

$$\lim_{\lambda \to \infty} P_{s/\lambda} f = f \ .$$

(b)(iii) Show that $\lambda R_\lambda \to I$ as $\lambda \to \infty$.

A: For every $f \in C$,

$$\lambda R_\lambda f = \int_0^\infty \lambda e^{-\lambda t} P_t f \,\mathrm{d}t = \int_0^\infty e^{-s} P_{s/\lambda} f \,\mathrm{d}s \ .$$

For every $s \geq 0$, by strong continuity of P_t

$$\lim_{\lambda \to \infty} P_{s/\lambda} f = f \ .$$

Therefore, by the dominated convergence theorem

$$\lim_{\lambda \to \infty} \lambda R_\lambda f = \int_0^\infty e^{-s} f \,\mathrm{d}s = f \ .$$

(c)(i) Using the definition of the infinitesimal generator and the fact that P_t and R_λ commute, show that

$$(\lambda - A)R_\lambda f = f, \quad \forall f \in \mathcal{D}(A).$$

(c)(i) Using the definition of the infinitesimal generator and the fact that P_t and R_λ commute, show that

$$(\lambda - A)R_\lambda f = f, \quad \forall f \in \mathcal{D}(A).$$

First part of proof of Corollary 2.17.

(c)(ii) Hence show that, for all $f \in \mathcal{D}(A)$,

$$\lambda(\lambda R_\lambda - I)f \to Af,$$

as $\lambda \to \infty$.

(c)(ii) Hence show that, for all $f \in \mathcal{D}(A)$,

$$\lambda(\lambda R_\lambda - I)f \to Af,$$

as $\lambda \to \infty$.

A: By a change of variable as in part (b)(iii)

$$\lambda(\lambda R_\lambda f - f) = \int_0^\infty \lambda e^{-s}(P_{s/\lambda} f - f)\,\mathrm{d}s = \int_0^\infty e^{-s} s \frac{\lambda}{s}(P_{s/\lambda} f - f)\,\mathrm{d}s\ .$$

(c)(ii) Hence show that, for all $f \in \mathcal{D}(A)$,

$$\lambda(\lambda R_\lambda - I)f \to Af,$$

as $\lambda \to \infty$.

A: By a change of variable as in part (b)(iii)

$$\lambda(\lambda R_\lambda f - f) = \int_0^\infty \lambda e^{-s}(P_{s/\lambda}f - f)\,\mathrm{d}s = \int_0^\infty e^{-s} s \frac{\lambda}{s}(P_{s/\lambda}f - f)\,\mathrm{d}s\ .$$

Denote $H(s,\lambda) = \frac{\lambda}{s}(P_{s/\lambda}f - f)$. Since $f \in \mathcal{D}(A)$, for every $s > 0$

$$\lim_{\lambda\to\infty} H(s,\lambda) = Af\ .$$

(c)(ii) **A: (continued)** There exists $\epsilon > 0$ such that for all $t \in (0, \epsilon)$

$$\frac{1}{t}\|P_t f - f\| \leq \|Af\| + 1 \; .$$

(c)(ii) **A: (continued)** There exists $\epsilon > 0$ such that for all $t \in (0, \epsilon)$

$$\frac{1}{t}\|P_t f - f\| \leq \|Af\| + 1 \ .$$

Then

- for $s/\lambda \in (0, \epsilon)$, $\|H(s, \lambda)\| \leq \|Af\| + 1$.

(c)(ii) **A: (continued)** There exists $\epsilon > 0$ such that for all $t \in (0, \epsilon)$

$$\frac{1}{t}\|P_t f - f\| \leq \|Af\| + 1 \ .$$

Then

- for $s/\lambda \in (0, \epsilon)$, $\|H(s, \lambda)\| \leq \|Af\| + 1$.
- For $s/\lambda \geq \epsilon$, P_t is a contraction, $\|H(s, \lambda)\| \leq \epsilon^{-1} 2\|f\|$.

(c)(ii) **A: (continued)** There exists $\epsilon > 0$ such that for all $t \in (0, \epsilon)$

$$\frac{1}{t}\|P_t f - f\| \leq \|Af\| + 1 \ .$$

Then

- for $s/\lambda \in (0, \epsilon)$, $\|H(s, \lambda)\| \leq \|Af\| + 1$.
- For $s/\lambda \geq \epsilon$, P_t is a contraction, $\|H(s, \lambda)\| \leq \epsilon^{-1} 2\|f\|$.

Hence, for all $s, \lambda > 0$

$$H(s, \lambda) \leq \epsilon^{-1} 2\|f\| + \|Af\| + 1 \ .$$

(c)(ii) **A: (continued)** There exists $\epsilon > 0$ such that for all $t \in (0, \epsilon)$

$$\frac{1}{t}\|P_t f - f\| \leq \|Af\| + 1 \ .$$

Then

- for $s/\lambda \in (0, \epsilon)$, $\|H(s, \lambda)\| \leq \|Af\| + 1$.
- For $s/\lambda \geq \epsilon$, P_t is a contraction, $\|H(s, \lambda)\| \leq \epsilon^{-1} 2\|f\|$.

Hence, for all $s, \lambda > 0$

$$H(s, \lambda) \leq \epsilon^{-1} 2\|f\| + \|Af\| + 1 \ .$$

Thus by dominated convergence,

$$\lim_{\lambda\to\infty} \lambda(\lambda R_\lambda f - f) = \int_0^\infty e^{-s} sAf \, \mathrm{d}s = Af .$$