C8.2 Stochastic Analysis and PDEs
2017 Q2

2.(a) Let b and σ be continuous functions on $\mathbb{R}$ and define $a = \sigma^2$.

Define what it means for a one-dimensional diffusion process X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ with generator

$$Af(x) = b(x)\frac{\partial f}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 f}{\partial x^2}$$

to satisfy

(i) the martingale problem for the operator A with initial point x;

Definition 3.2

(ii) the martingale problem $M(a, b)$.

Definition 3.11

(b) Now assume that there is a constant δ such that $a(x) > \delta > 0$. Show that, if the diffusion process X on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, satisfies $M(a, b)$, then X is a weak solution to the stochastic differential equation

$$X_t = X_0 + \int_0^t b(X_u)\,\mathrm{d}u + \int_0^t \sigma(X_u)\,\mathrm{d}B_u,$$

where B is a Brownian motion.

Theorem 3.12

(c) Now assume that there is a unique (in law) weak solution X to the stochastic differential equation

$$X_t = X_0 + \int_0^t \kappa(\theta - X_u)\mathrm{d}u + \int_0^t \sigma\sqrt{X_u}\mathrm{d}B_u, \tag{1}$$

where κ, θ, σ are strictly positive constants and B is a Brownian motion.

Give an integral expression for the scale function and give the density of the speed measure for this process.

(c) A: (Scale function)

Solution 1: From Definition 6.1, the scale function is given by

$$S(x) = \int_{x_0}^{x} e^{-\int_{y_0}^{y} 2\kappa(\theta - z)/(\sigma^2 z)\, \mathrm{d}z}\, \mathrm{d}y = \int_{x_0}^{x} \left(\frac{y}{y_0}\right)^{-2\kappa\theta/\sigma^2} e^{2\kappa(y-y_0)/\sigma^2}\, \mathrm{d}y$$

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Solution 2: The scale function satisfies $\kappa(\theta - x)S'(x) + \frac{1}{2}\sigma^2 x S''(x) = 0$. Denoting $u(x) = S'(x)$,

$$\frac{u'(x)}{u(x)} = -\frac{2\kappa}{\sigma^2}(\frac{\theta}{x} - 1) \Longrightarrow \log u(x) = -\frac{2\kappa\theta}{\sigma^2}\log x + \frac{2\kappa}{\sigma^2}x + C_0 \ .$$

Therefore $S'(x) = u(x) = e^{C_0} x^{-2\kappa\theta/\sigma^2} e^{2\kappa x/\sigma^2}$, hence

$$S(x) = \int_{x_0}^{x} e^{C_0} y^{-2\kappa\theta/\sigma^2} e^{2\kappa y/\sigma^2}\,\mathrm{d}y \ .$$

(c) A: (Speed density)

By definition,

$$m(x) = \frac{1}{S'(x)\sigma^2 x} = \frac{1}{\sigma^2 x} e^{C_0} x^{2\kappa\theta/\sigma^2} e^{-2\kappa x/\sigma^2} \ .$$

(d) For the diffusion X satisfying

$$X_t = X_0 + \int_0^t \kappa(\theta - X_u)\mathrm{d}u + \int_0^t \sigma\sqrt{X_u}\mathrm{d}B_u, \tag{2}$$

let $T_0 = \inf\{t \geq 0 : X_t = 0\}$.

By considering the exit probability from the interval (ϵ, b) as $\epsilon \to 0$ determine conditions on the parameters for which the process cannot reach the origin in that $P^x(T_0 < \infty) = 0$ for all $x > 0$.

(d) A: Let $T_a = \inf\{t \geq 0 : X_t = a\}$. Then

$$\mathbb{P}^x[T_\epsilon < T_b] = \frac{S(b) - S(x)}{S(b) - S(\epsilon)} = \frac{\int_x^b y^{-2\kappa\theta/\sigma^2} e^{2\kappa y/\sigma^2} \, \mathrm{d}y}{\int_\epsilon^b y^{-2\kappa\theta/\sigma^2} e^{2\kappa y/\sigma^2} \, \mathrm{d}y} .$$

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We have $\int_\epsilon^b y^{-2\kappa\theta/\sigma^2} e^{2\kappa y/\sigma^2} \, \mathrm{d}y \to_{\epsilon \to 0} \infty$ if and only if $2\kappa\theta/\sigma^2 \geq 1$.

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Therefore $\mathbb{P}^x[T_0 < T_b] = \lim_{\epsilon \to 0} \mathbb{P}^x[T_\epsilon < T_b] = 0$ if and only if $2\kappa\theta \geq \sigma^2$.

Therefore $\mathbb{P}^x[T_0 < \infty] = \mathbb{P}^x[\cup_{b=1}^\infty \{T_0 < T_b\}] = 0$ if and only if $2\kappa\theta \geq \sigma^2$.