C8.2 Stochastic Analysis and PDEs
2018 Q2

2. (a) Let X be a diffusion process taking values in $\mathbb{R}^d$ satisfying the stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t;$$

where $b = (b_i)_{i=1}^d, (\sigma_{ij})_{i,j=1}^d$ and $b_i, \sigma_{ij} \colon \mathbb{R} \to \mathbb{R}$ for $1 \leq i, j \leq d$ are Lipschitz continuous functions of linear growth and W is a d-dimensional Brownian motion. Let $a(x) = \sigma(x)\sigma^{\mathrm{T}}(x)$.

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(1) the martingale problem $M(a, b)$ for X.

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(a) (ii) For $d = 1$, show that a solution to the martingale problem for A is a solution to $M(a, b)$.

Lemma 3.13

(a) (iii) State conditions under which a sequence of discrete time Markov chains X^h with transition kernel $\pi_h(x, \mathrm{d}y)$, taking values in a sequence of state spaces $E^h \subset \mathbb{R}^d$, will converge weakly to a diffusion process X satisfying the martingale problem $M(a, b)$, taking values in $\mathbb{R}^d$ as $h \to 0$.

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Theorem 4.12 **A:** Define $K_h(x, \mathrm{d}y) = \frac{1}{h}\pi_h(x, \mathrm{d}y)$ and for $x \in E^h$, $1 \leq i, j \leq d$, $\epsilon > 0$,

$$a_{ij}^h(x) = \int_{|y-x|\leq 1} (y_i - x_i)(y_j - x_j) K_h(x, \mathrm{d}y)$$

$$b_i^h(x) = \int_{|y-x|\leq 1} (y_i - x_i) K_h(x, \mathrm{d}y)$$

$$\Delta_\epsilon^h(x) = K_h(x, B^c(x, \epsilon)).$$

(a) (iii) **A: (continued)**

Suppose that

(1) for all $R > 0$, $\epsilon > 0$, and $1 \leq i, j \leq d$

$$\lim_{h \to 0} \sup_{|x| \leq R} |a_{ij}^h(x) - a_{ij}(x)| + |b_i^h(x) - b_i(x)| + \Delta_\epsilon^h(x) = 0 \ ,$$

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(2) and $X_0^h \to x_0$.

Then $X^h \to_{h \to 0} X$ weakly.

2. (b) A discrete time simple random walk $Y^h = \{Y_n^h : n \in \mathbb{N}\}$ on $E_h = \sqrt{h}\mathbb{Z}^d$ has transition probabilities

$$P(Y_{n+1}^h = x \pm \sqrt{h} k e_i \mid Y_n^h = x) = \frac{1}{2d} p(k), \quad \forall x \in E_h,$$

where e_i are unit vectors in the e_i direction and $\{p(k) : k \in \mathbb{Z}_+\}$ is a probability distribution on $\mathbb{Z}_+$ with mean 1; finite variance

$$\sigma^2 = \sum_{k=0}^{\infty} (k-1)^2 p(k)$$

and finite fourth moment.

Show that Y^h will converge weakly to a suitable constant time change of Brownian motion in $\mathbb{R}^d$.

(b) **A:** Let Z be random variable on $\mathbb{Z}_+$ with law p. Then Y^h is the symmetric random walk on $\sqrt{h}\mathbb{Z}^d$ with step size $\sim \sqrt{h}Z$.

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Applying (a)(iii), $Y_t^h \to_{h \to 0} \sqrt{\frac{1}{d}(\sigma^2+1)} B_t \sim B_{\frac{1}{d}(\sigma^2+1)t}$.

(c) Now take $d = 1$ and consider a random walk in which

$$P(Y^h_{n+1} = x + \sqrt{h} \mid Y^h_n = x) = \frac{1}{2}(1 + \mu h^\alpha),$$

$$P(Y^h_{n+1} = x - \sqrt{h} \mid Y^h_n = x) = \frac{1}{2}(1 - \mu h^\alpha),$$

where μ, α are positive constants.

(i) Let $\alpha = 1/2$. Show that the limiting process is a Brownian motion with drift.

(c)(i) **A:** In the notation of (a)(iii), since $\alpha = \frac{1}{2}$,

$$b^h(x) = \frac{1}{h}\left(\sqrt{h}\frac{1}{2}\left(1+\mu\sqrt{h}\right) - \sqrt{h}\frac{1}{2}\left(1-\mu\sqrt{h}\right)\right) = \mu$$

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Applying (a)(iii), $X_t^h \to_{h\to 0} \mu t + B_t$.

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(i) $\alpha < 1/2$, then $b^h(x) \to \infty$ and thus X^h will not converge. However $h^{1/2-\alpha} X_t^h \to \mu t$, uniform motion to the right.