C8.2 Stochastic Analysis and PDEs
2019 Q3

3.(a) Let $D(0, r)$ be the ball of radius r about the origin in $\mathbb{R}^2$. Let B be a Brownian motion in $\mathbb{R}^2$ and consider the sequence of stopping times defined by $T_0 = 0$ and for all $k \geq 1$

$$\begin{aligned} S_k &= \inf\{t > T_{k-1} : B_t \in D(0, r)\}, \\ T_k &= \inf\{t > S_k : B_t \notin D(0, 2r)\}. \end{aligned}$$

(i) Show that $\mathbb{E}^0 T_1 < \infty$ and hence that the random variables T_k are finite almost surely.

(a)(i) **A:** Since $B_1 - B_0$ is a Gaussian random variable, for all $x \in \mathbb{R}^2$

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Hence,

$$\begin{aligned}
\mathbb{P}^x[T_1 > n] &\leq \mathbb{P}^x[\cap_{k=1}^n \{|B_k - B_{k-1}| \leq 4r\}] \\
&= \prod_{k=1}^n \mathbb{P}^x[|B_k - B_{k-1}| \leq 4r] \quad \text{(independence of increments)} \\
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Therefore

$$\mathbb{E}^x[T_1] = \int_0^\infty \mathbb{P}^x[T_1 > r]\,\mathrm{d}r \leq \sum_{n=0}^\infty \mathbb{P}^x[T_1 > n] < \infty \ .$$

(a)(i) **A: (continued)** To show $T_k < \infty$ a.s. for all $k \geq 1$, recall that for $\tau_r = \inf\{t \geq 0 : |B_t| = r\}$ and $r \leq |x| \leq R$

$$\mathbb{P}^x[\tau_r < \tau_R] = \frac{\log R - \log |x|}{\log R - \log r} \ .$$

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By the strong Markov property, $\mathbb{P}[S_k < \infty \mid B_{T_{k-1}}] = 1$, and by the previous part $\mathbb{P}[T_k < \infty \mid B_{S_k}] = 1$.

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A: The random variables $\int_{S_k}^{T_k} I_{D(0,r)}(B_t)\,\mathrm{d}t$ are

- clearly positive (i.e. non-negative),
- independent by strong Markov property,
- identically distributed by rotation invariance.

(a)(iii) Let U be a bounded domain in $\mathbb{R}^2$ and let $x \in \mathbb{R}^2$. Show that the occupation time of the set U is infinite in that

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Moreover, $\int_{S_k}^{T_k} I_{D(0,r)}(B_t)\,\mathrm{d}t$ is not a.s. zero, hence

$$\mathbb{E}[\int_{S_k}^{T_k} I_{D(0,r)}(B_t)\,\mathrm{d}t] > 0\ .$$

(a)(iii) **A: (continued)** Therefore, by the strong law of large numbers

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_{S_k}^{T_k} I_{D(0,r)}(B_t)\,\mathrm{d}t > 0 \; .$$

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Thus, a.s. $\sum_{k=1}^{\infty} \int_{S_k}^{T_k} I_{D(0,r)}(B_t)\,\mathrm{d}t = \infty$ and

$$\int_0^\infty I_U(B_t)\,\mathrm{d}t \geq \sum_{k=1}^{\infty} \int_{S_k}^{T_k} I_{D(0,r)}(B_t)\,\mathrm{d}t = \infty \ .$$

(b) Let $u : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + g(t,x)u, \quad x \in \mathbb{R}^2,$$

$$u(x,0) = u_0(x), \quad \forall x \in \mathbb{R}^2,$$

where u_0 is a bounded integrable function from $\mathbb{R}^2$ to $\mathbb{R}$ and
$g : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a bounded integrable function.

Prove that if there is a solution u to the PDE, which is bounded on compact time intervals, then it can be expressed in terms of Brownian motion as

$$u(t,x) = \mathbb{E}^x \left(\exp\Big(\int_0^t g(t-s, X_s)\,\mathrm{d}s\Big) u_0(X_t) \right).$$

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(c) Let $v : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the PDE

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - \alpha x \cdot \nabla v + g(x)v + h(x), \quad x \in \mathbb{R}^2,$$

$$v(x, 0) = v_0(x), \quad \forall x \in \mathbb{R}^2,$$

where g, h, v_0 are bounded integrable functions from $\mathbb{R}^2$ to $\mathbb{R}$ and α is a constant.

(i) Derive a representation for $v(t, x)$ in terms of a suitable diffusion process X.

(c)(i) **A:** For Brownian motion $W\colon [0,\infty) \to \mathbb{R}^2$, consider the solution to

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For $t \geq 0$ fixed and $0 \leq s \leq t$ define

$$M_s^t = v(t-s, X_s) e^{\int_0^s g(X_u)\,\mathrm{d}u} + \int_0^s h(X_r) e^{\int_0^r g(X_u)\,\mathrm{d}u}\,\mathrm{d}r \ .$$

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By Itô's formula, denoting $Y_s = e^{\int_0^s g(X_u)\,\mathrm{d}u}$,

$$\begin{aligned}
\mathrm{d}M_s^t = Y_s \Big(&- \frac{\partial v}{\partial s}(t-s, X_s)\,\mathrm{d}s + \sum_{i=1}^{2} \frac{\partial v}{\partial x_i}(t-s, X_s)\,\mathrm{d}X_s^i \\
&+ \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 v}{\partial x_i \partial x_j}(t-s, X_s)\,\mathrm{d}\langle X^i, X^j \rangle_s \\
&+ v(t-s, X_s) g(X_s)\,\mathrm{d}s + h(X_s)\,\mathrm{d}s \Big)
\end{aligned}$$

(c)(i) **A: (continued)** Since v solves the given PDE,

$$\mathrm{d}M_s^t = Y_s \sum_{i=1}^{2} \frac{\partial v}{\partial x}(t-s, X_s)\,\mathrm{d}W_s^i \ .$$

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Hence, by continuity of v,

$$\begin{aligned} v(t,x) &= \mathbb{E}[M_0^t \mid X_0 = x] = \mathbb{E}[M_t^t \mid X_0 = x] \\ &= \mathbb{E}\Big[v_0(X_t)e^{\int_0^t g(X_u)\,\mathrm{d}u} + \int_0^t h(X_r)e^{\int_0^r g(X_u)\,\mathrm{d}u}\,\mathrm{d}r \mid X_0 = x\Big] \ . \end{aligned}$$

(c)(ii) Let $C_t = \int_0^t I_{D(0,1)}(X_s)ds$ and let $\theta \geq 0$. Find a PDE for $L(t,x) = \mathbb{E}^x\left(\exp(-\theta C_t)\right)$, the Laplace transform of the occupation time of the unit ball in $\mathbb{R}^2$ up to time t by the diffusion process X.

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A: Take $g = -\theta I_{D(0,1)}$, $h = 0$, and $v_0 = 1$.

By part (ii), the solution to

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - \alpha x \cdot \nabla v + -\theta I_{D(0,1)}(x) v, \quad x \in \mathbb{R}^2,$$

$$v(x,0) = 1, \quad \forall x \in \mathbb{R}^2,$$

is given by

$$v(t,x) = \mathbb{E}\Big[\exp\Big(-\theta \int_0^t I_{D(0,1)}(X_s)\,\mathrm{d}s\Big)\Big] = L(t,x)\ .$$

Therefore $L(t,x)$ solves the same PDE as v.