

Finite Element Methods. QS 2

Hand-in deadline: 9.00 am, 17 Feb, week 5.

Class: 19 Feb, week 5.

1. Why is

$$u(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

not in $W^{1,p}(-1, 1)$ for any $1 \leq p \leq \infty$?

2. Consider the problem

$$-u'' - k^2 u = f, \quad u(0) = 0 = u(\pi)$$

where $f \in L^2(\Omega)$ and $k^2 \in \mathbb{R}$.

- (a) Cast the problem in variational form, stating carefully the spaces employed.
- (b) For what values of k is the problem not well-posed? (Hint: take $f = 0$ and look for nonzero solutions.)
- (c) For small values of k , the problem is coercive. For large values of k , the problem is not coercive. For what value of k does the problem lose coercivity?

3. Consider the variational formulation: find $u \in H^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \alpha \int_{\partial\Omega} uv \, ds = \int_{\Omega} v f \, dx \quad \text{for all } v \in H^1(\Omega),$$

where $f \in L^2(\Omega)$ and real $\alpha > 0$.

- (a) Show that a solution u satisfying the Poisson equation $-\nabla^2 u = f$ in Ω and a Robin condition $\alpha u + \frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$ also satisfies the above weak formulation.
- (b) Show that $c(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \alpha \int_{\partial\Omega} uv \, ds$ defines an inner product over $H^1(\Omega)$, and hence establish that a solution of the weak formulation is uniquely defined.
4. Let $a(u, v) = \int_0^1 (u'v' + u'v + uv) \, dx$ and $V = \{v \in H^1(\Omega) \mid v(0) = v(1) = 0\}$. Show that $a(v, v) = \int_0^1 [(v')^2 + v^2] \, dx$ for all $v \in V$. (Hint: write $vv' = \frac{1}{2}(v^2)'$.)

5. (a) Suppose that Ω has boundary Γ , and that $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

Consider the quadratic functional $J : v \in H^1(\Omega) \mapsto J(v) \in \mathbb{R}$ defined by

$$J(v) = \frac{1}{2} \int_{\Omega} (|\nabla v|^2 + v^2) \, dx + \frac{1}{2} \int_{\Gamma} v^2 \, ds - \int_{\Omega} f v \, dx - \int_{\Gamma} g v \, ds.$$

(i) Show that if $u \in H^1(\Omega)$ is such that $J(u) \leq J(v)$ for all $v \in H^1(\Omega)$, then there exist a bilinear functional $a(\cdot, \cdot)$ defined on $H^1(\Omega) \times H^1(\Omega)$ and a linear functional $F(\cdot)$ defined on $H^1(\Omega)$, such that

$$a(u, v) = F(v) \quad \forall v \in H^1(\Omega). \quad (\text{P})$$

(ii) State the strong form of the PDE.

(b)

(i) Given a subspace $V_h \subset H^1(\Omega)$, state the Galerkin approximation of P.

(ii) Show that the Galerkin approximation is equivalent to a linear system, and give formulae for the associated matrix and vector.

(iii) Show that the associated linear system has a unique solution u_h .

(iv) Show that $J(u) \leq J(u_h) \leq J(v_h)$ for all $v_h \in V_h$.

6. Let $\Omega = (0, 1)^2$ with boundary $\partial\Omega$. Let $\Gamma_D = \{(x, y) \in \partial\Omega : x = 0\}$, and let $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Consider the boundary value problem

$$\begin{aligned} -\nabla^2 u + 2u + \sin(x)u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \nabla u \cdot n &= 0 && \text{on } \Gamma_N, \end{aligned}$$

where n denotes the outward unit normal to Γ_N .

Write this boundary value problem as:

- (i) a variational problem over a function space V , to be defined;
- (ii) the minimisation of an energy functional $J : V \rightarrow \mathbb{R}$;
- (iii) an equation over elements of V^* , carefully introducing any necessary operators;
- (iv) an equation over elements of V .

7. (a) Formulate the following differential equation as a variational problem on $H_0^1(0, 1)$:

$$-u'' + u' + u = f, \quad u(0) = 0 = u(1).$$

(b) Show that the bilinear form from this variational problem is coercive and bounded.

(c) State and prove Céa's Lemma.

(d) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0, 1]$ with maximum width h . Let u_h be the solution to the Galerkin approximation of the variational problem using V_h , and let $I_h : H^2(0, 1) \rightarrow V_h$ be the interpolation operator onto V_h . Assuming $u \in H^2(0, 1)$, and the following result,

$$\|u - I_h u\|_{H^1(0,1)} \leq h \|u''\|_{L^2(0,1)},$$

show that

$$\|u - u_h\| \leq Dh \|u''\|_{L^2(0,1)},$$

and provide a numerical value for D .