Finite Element Methods. QS 3

Hand-in deadline: 9.00 am, 2 Mar, week 7.
Class: 4 Mar, week 7.

1. Let $K = [0, 1]$, $\mathcal{V} = \mathcal{P}_3(K)$, and let $\mathcal{L} = \{\ell_0, \ell_1, \ell_0^*, \ell_1^*\}$, where

$$\begin{aligned} \ell_0 &: v \mapsto v(0), \\ \ell_1 &: v \mapsto v(1), \\ \ell_0^* &: v \mapsto v'(0), \\ \ell_1^* &: v \mapsto v'(1). \end{aligned}$$

This defines the first Hermite finite element in one dimension.

(a) Show that $\mathcal{L}$ determines $\mathcal{V}$.

(b) Construct the nodal basis $\{\phi_0, \phi_1, \phi_0^*, \phi_1^*\}$ for this finite element. Express your answers in the monomial basis.

(c) Consider the problem

$$-u'' + u = f, \quad u'(0) = 0 = u'(1).$$

Discretise $\Omega = [0, 1]$ into N intervals of uniform mesh size $h = 1/N$ and let V_h be the function space constructed by equipping each cell with the Hermite finite element defined above. This induces a linear system

$$Ax = b.$$

(i) State formulae for the components of A and b.
(ii) For a given cell K, what is the local 4×4 matrix A_K of contributions to A?
(iii) Apply the finite element assembly algorithm cellwise to construct the matrix A for the case $N = 3$.

2. Consider the problem

$$-Tu'' = -\rho g, \quad u(0) = 0 = u(L),$$

with $L = 10$ m, $\rho = 1$ kg/m, $g = 9.8$ ms^{-2}, and $T = 98$ N. Here u is the vertical deflection of a hanging cable sagging under gravity. Compute the Galerkin approximation to this problem using the trial space

$$V_h = \text{span}\{\sin\left(\frac{\pi x}{L}\right), 1 - \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right)\}.$$

Note that each basis function of V_h satisfies the boundary conditions, and hence $u_h \in V_h$ will also. (Note: it may be more convenient to use a symbolic algebra package such as sympy, maple or mathematica for this.)

3. A quadrature rule on a cell K has degree of precision m if it computes the exact answer for all polynomials of degree m or less.

What is the minimum degree of precision required to exactly assemble the matrix for the following bilinear forms? Furthermore, in one dimension, how many quadrature points n are required when using Gaussian quadrature?

(a) quadratic Lagrange elements, constant coordinate transformation Jacobian $J_K(\hat{x})$. Form:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x.$$

(b) quintic Lagrange elements, linear coordinate transformation Jacobian $J_K(\hat{x})$. Form:

$$a(u, v) = \int_\Omega uv \, \mathrm{d}x.$$

4. Let K be a non-degenerate triangle, and let $\mathcal{V} = \mathcal{P}_2(K)$. Denote the vertices of K by z_i, $i = 1, \ldots, 3$. Let

$$\begin{aligned}
\ell_1 &: v \mapsto v(z_1), \\
\ell_2 &: v \mapsto v(z_2), \\
\ell_3 &: v \mapsto v(z_3), \\
\ell_4 &: v \mapsto v(\frac{z_1 + z_2}{2}), \\
\ell_5 &: v \mapsto v(\frac{z_1 + z_3}{2}), \\
\ell_6 &: v \mapsto v(\frac{z_2 + z_3}{2}).
\end{aligned}$$

Show that $\mathcal{L} = \{\ell_1, \ldots, \ell_6\}$ determines $\mathcal{V}$.

5. In lectures we saw that the Argyris element is the $C^1(\Omega)$-conforming finite element of lowest degree. Its practical implementation is greatly complicated by the following fact.

Define the reference triangle $\hat{K}$ to be the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$. Show by counterexample or otherwise that if an affine map is used for the coordinate transformation

$$x = F_K(\hat{x})$$

from $\hat{K}$ to an arbitrary physical triangle K, then the normals to the edges of $\hat{K}$ are not preserved in general.

Comment: This has the unfortunate consequence that the reference basis functions corresponding to the midpoint normal flux degrees of freedom are not in general mapped to the corresponding basis functions on the physical triangle under an *affine* map. (More complicated maps must be used.)