Finite Element Methods. QS 4

This sheet is not to be turned in. Complete it, and check your answers with the provided solutions.
Class: Trinity term.

1. Consider the problem:

$$u = \underset{v \in H^1(\Omega)}{\operatorname{argmin}} \; J(v)$$

where

$$J(v) = \frac{1}{2} \int_\Omega \gamma \nabla v \cdot \nabla v + \frac{1}{2} \left(v^2 - 1\right)^2 \, \mathrm{d}x.$$

(i) State the Euler–Lagrange equation that characterises stationary points u of this functional.
(ii) Given an initial guess u_0, state the linearised system of equations that must be solved for the update δu in a Newton–Kantorovich iteration, in weak form.

2. Consider the stationary incompressible isothermal Newtonian Navier–Stokes equations, given in strong form by

$$\begin{aligned} -\nabla^2 u + (u \cdot \nabla) u + \nabla p &= f \text{ in } \Omega, \\ \nabla \cdot u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Write this system of equations in weak form as a nonlinear variational equation.

Write the linearised system of equations that must be solved for the update $(\delta u, \delta p)$ in a Newton–Kantorovich iteration, in weak and strong form.

3. Let V and Y be Banach spaces.

Consider the Newton–Kantorovich iteration applied to
(i) $F : V \to V^*$;
(ii) $(G \circ F) : V \to Y$, where $G : V^* \to Y$ is linear, continuous and invertible.

Prove that the Newton–Kantorovich iteration yields the same sequence of iterates in both cases when initialised from the same initial guess $u_0 \in V$.

Remark: this is the property of *affine covariance*, and is fundamental to the proper understanding of Newton-type methods. For a full discussion of this property, see P. Deuflhard, *Newton Methods for Nonlinear Problens*, Springer-Verlag, 2011.

4. Given a Hilbert space V, a bilinear form $a : V \times V \to \mathbb{R}$ and a linear form $F : V \to \mathbb{R}$, consider the problem: find $u \in V$ such that

$$a(u, v) = F(v) \text{ for all } v \in V. \tag{V}$$

(i) State the *Riesz representation theorem*. Explain briefly in what cases it guarantees the well-posedness of (V).
(ii) State the *Lax–Milgram* theorem regarding the well-posedness of (V).
(iii) State *Babuška's theorem* regarding the well-posedness of (V).
(iv) Show that if (V) is well-posed by the Riesz representation theorem, then it satisfies the conditions of the Lax–Milgram theorem.
(v) Show that if (V) satisfies the conditions of the Lax–Milgram theorem, then it satisfies the conditions of Babuška's theorem.

5. Let $V = H_0^1(\Omega; \mathbb{R}^n)$ and $Q = L_0^2(\Omega)$. Let

$$L(u, p) = \frac{1}{2} \int_\Omega \nabla u : \nabla u \, \mathrm{d}x - \int_\Omega f \cdot u \, \mathrm{d}x - \int_\Omega p \nabla \cdot u \, \mathrm{d}x.$$

We say (u, p) is a saddle point of L iff

$$L(u, q) \le L(u, p) \le L(v, p)$$

for all $v \in V$, $q \in Q$.

Show that (u, p) is a weak solution of the Stokes equations if and only if it is a saddle point of the Lagrangian.

Hint: consider the two inequalities separately.

6. Consider the (contrived) problem

$$\begin{aligned} u = \underset{v \in H^1(\Omega; \mathbb{R}^3)}{\operatorname{argmin}} \quad & \frac{1}{2} \int_\Omega \nabla v : \nabla v + v^2 \, \mathrm{d}x - \int_\Omega f \cdot v \, \mathrm{d}x, \\ \text{subject to} \quad & \nabla \times v = 0. \end{aligned}$$

Suppose one were to discretise the Euler-Lagrange system for this problem with piecewise linear Lagrange elements for every variable. Could this discretisation be stable? Why or why not?

7. (Advanced, optional)

Let $Z : V \to V$ be a continuous, invertible operator. Suppose that it is a symmetry of a residual $F : V \to V^*$, i.e. it satisfies

$$ZRF(u) = RF(Zu)$$

for all $u \in V$, where $R : V^* \to V$ is the Riesz map. (Imagine, for example, that Z were a reflection; this property asserts that the reflection of the residual is the residual of the reflection.)

Prove that if the Newton–Kantorivich iteration is initialised from an initial guess u_0 that satisfies $Zu_0 = u_0$, then all subsequent iterates u_k will also satisfy $Zu_k = u_k$, so long as the iterates are defined.

Remark: this means that one should be careful when solving nonlinear problems with symmetries. In general, such problems will support both symmetric and nonsymmetric solutions. If the initial guess is chosen to be symmetric, then only symmetric solutions will be found by the iteration if the linear subproblem is solved exactly.