Finite Element Methods. QS 4

1. (a) Minimizer when Fréchet Derivative equal to zero:

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{J(u+\epsilon v)-J(u)}{\epsilon} &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\Big(\int_\Omega \gamma\nabla(u+\epsilon v)\cdot\nabla(u+\epsilon v)+\frac{1}{2}\left((u+\epsilon v)^2-1\right)^2\ \mathrm{d}x \\ &\quad -\int_\Omega \gamma\nabla u\cdot\nabla u+\frac{1}{2}\left(u^2-1\right)^2\ \mathrm{d}x\Big) \\ &= \int_\Omega \gamma(\nabla u\cdot\nabla v+u^3v-uv)\,\mathrm{d}x=0.\end{aligned}$$

Weak form: find $u\in H^1(\Omega)$ such that

$$G(u;v)=\int_\Omega(\gamma\nabla u\cdot\nabla v+u^3v-uv)\,\mathrm{d}x=0 \quad \text{for all } v\in H^1(\Omega)$$

Strong form:

$$-\gamma\nabla^2u+u^3-u=0 \text{ in } \Omega, \quad \nabla u\cdot n=0 \text{ on } \partial\Omega.$$

(b) Taking the Fréchet derivative of G with respect to u, we find that the linearisation at a fixed u in the direction of $w\in H^1(\Omega)$ is

$$G_u(u;v,w)=\int_\Omega \gamma(\nabla w\cdot\nabla v)+3u^2wv-wv)\,\mathrm{d}x.$$

Thus, the Newton update solves: find $\delta u\in H^1(\Omega)$ such that

$$G_u(u;v,\delta u)=-G(u;v) \quad \text{for all } v\in H^1(\Omega).$$

2.

Let $V=H_0^1(\Omega;\mathbb{R}^n)$ and $Q=L_0^2(\Omega)$. The weak form I'm looking for is the following: find $(u,p)\in V\times Q$ such that

$$\begin{aligned}\int_\Omega \nabla u:\nabla v\ \mathrm{d}x+\int_\Omega(u\cdot\nabla u)\cdot v\ \mathrm{d}x-\int_\Omega p(\nabla\cdot v)\ \mathrm{d}x&=\int_\Omega f\cdot v\ \mathrm{d}x,\\ -\int_\Omega q(\nabla\cdot u)\ \mathrm{d}x&=0,\end{aligned}$$

for all $(v,q)\in V\times Q$.

After calculating the Gâteaux derivative, the linearised problem is: find $(\delta u,\delta p)\in V\times Q$ such that

$$\begin{aligned}\int_\Omega \nabla\delta u:\nabla v\ \mathrm{d}x+\int_\Omega(\delta u\cdot\nabla u)\cdot v\ \mathrm{d}x+\int_\Omega(u\cdot\nabla\delta u)\cdot v\ \mathrm{d}x-\int_\Omega \delta p(\nabla\cdot v)\ \mathrm{d}x&=R_v(u,p,v),\\ -\int_\Omega q(\nabla\cdot\delta u)\ \mathrm{d}x&=R_q(u,v,q),\end{aligned}$$

for all $(v, q) \in V \times Q$. Here R_v and R_q represent the residuals, the two equations above with all terms taken to the LHS.

In strong form, this becomes

$$\begin{aligned} -\nabla^2 \delta u + (\delta u \cdot \nabla) u + (u \cdot \nabla) \delta u + \nabla \delta p &= R_v \text{ in } \Omega, \\ \nabla \cdot \delta u &= R_p \text{ in } \Omega, \\ \delta u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

3.

Let $H(u) = GF(u)$. Then $H_u(u; \delta u) = GF_u(u; \delta u)$ by the definition of the Fréchet derivative and linearity of G. Newton-Kantorovich iteration:

(i) Find $\delta u \in V$ such that $F_u(u; \delta u) = -F(u)$.

(ii) Find $\delta u \in V$ such that $H_u(u; \delta u) = -H(u) \Leftrightarrow GF_u(u; \delta u) = -GF(u) \Leftrightarrow F_u(u; \delta u) = -F(u)$ since G is invertible.

So iterations are the same if starting with same initial guess.

4.

(i)
Theorem (Riesz Representation Theorem)**.** *Let X be a Hilbert Space. Any bounded linear functional $j \in X^*$ can be uniquely represented by a $g \in X$, via*

$$\langle j, u \rangle = (g, u).$$

Moreover, the norms agree: $\|j\|_{X^*} = \|g\|_X$.

Well-posedness is guaranteed by a being an inner product, i.e. symmetric continuous and coercive.

(ii)
Theorem (Lax-Milgram)**.** *Let V be a closed subspace of a Hilbert space H. Let $a : H \times H \to \mathbb{R}$ be a (not necessarily symmetric) continuous coercive bilinear form, and let $F \in V$. Consider the variational problem:*

$$\textit{find } u \in V \textit{ such that } a(u, v) = F(v) \textit{ for all } v \in V.$$

This problem has a unique stable solution.

Well-posedness is guaranteed by a being continuous and coercive.

(iii)
Theorem (Babuškas theorem: necessary and sufficient conditions)**.** *Let V_1 and V_2 be two Hilbert spaces with inner products $(\cdot, \cdot)_{V_1}$ and $(\cdot, \cdot)_{V_2}$ respectively. Let $a : V_1 \times V_2 \to \mathbb{R}$ be a bilinear form for which there exist constants $C < \infty$, $\gamma > 0$, $\gamma' > 0$ such that*

1. $|a(u, v)| \leq C \|u\|_{V_1} \|v\|_{V_2}$;

2. $\gamma \leq \inf_{\substack{u \in V_1 \\ u \neq 0}} \sup_{\substack{v \in V_2 \\ v \neq 0}} \frac{a(u,v)}{\|u\|_{V_1} \|v\|_{V_2}}$;

3. $\gamma' \leq \inf_{\substack{v \in V_2 \\ v \neq 0}} \sup_{\substack{u \in V_1 \\ u \neq 0}} \frac{a(u,v)}{\|u\|_{V_1} \|v\|_{V_2}}$;

for all $u \in V_1$, $v \in V_2$. Then for all $F \in V_2^*$ there exists exactly one element $u \in V_1$ such that

$$a(u,v) = F(v) \text{ for all } \in V_2.$$

Furthermore the problem is stable in that

$$\|u\|_{V_1} \leq \frac{\|F\|_{V_2^*}}{\gamma}.$$

Well-posedness is guaranteed by a being continuous and satifying the inf sup conditions.

(iv) Well-posed by the Riesz Representation Theorem $\Rightarrow$ a is symmetric, continuous, and coercive, so it satisfies the conditions of Lax-Milgram (continuous and coercive).

(v) For Lax-Milgram $V_1 = V_2 = V$. Well-posed under Lax-Milgram $\Rightarrow$ a is continuous and coercive.

Continuity implies the first condition of Babuška.

Coercivity implies $\frac{a(u,u)}{\|u\|_V^2} \geq \gamma''$ for all $u \in V$, for some $\gamma'' > 0$. The inf-sup conditions are easily satisfied.

5. Let

$$a(u,v) = \int_\Omega \nabla u : \nabla v \, \mathrm{d}x, \quad b(v,q) = -\int_\Omega q \nabla \cdot v \, \mathrm{d}x, \quad f(v) = \int_\Omega f \cdot v \, \mathrm{d}x.$$

We have that $L(u,p) = \frac{1}{2}a(u,u) + b(u,p) - f(u)$. Start with first inequality:

$$\begin{aligned} \forall q \in Q,\ L(u,q) \leq L(u,p) &\Leftrightarrow \forall q \in Q,\ L(u,q) - L(u,p) \leq 0 \\ &\Leftrightarrow \forall q \in Q,\ b(u,q) - b(u,p) \leq 0 \\ &\Leftrightarrow \forall q \in Q,\ b(u,q-p) \leq 0 \\ &\Leftrightarrow \forall q \in Q,\ b(u,q) = 0, \end{aligned}$$

where in the last step we used the fact that Q is a vector space (take $q+p$ and $-q+p$).

Recall that if a is symmetric and positive. Then u solves $a(u,v) = f(v)$ for all $v \in V$ if and only if u minimises $J(v) = \frac{1}{2}a(v,v) - f(v)$ in V. Let $J_p(v) = \frac{1}{2}a(v,v) + b(v,p) - f(v)$. Second inequality:

$$\begin{aligned} \forall v \in V,\ L(u,p) \leq L(v,p) &\Leftrightarrow u \text{ minimises } J_p \text{ in } V \\ &\Leftrightarrow \forall v \in V,\ a(u,v) + b(v,p) = f(v). \end{aligned}$$

(the last line is the weak form of Stokes)

6.

Let the Lagrangian

$$L(v,q) = \frac{1}{2}\int_\Omega \left(\nabla v : \nabla v + v^2\right) \mathrm{d}x - \int_\Omega f \cdot v \,\mathrm{d}x - \int_\Omega q \cdot \nabla \times v \,\mathrm{d}x,$$

where q is a vector function.

We can find the Euler-Lagrange equations for this problem by taking the Fréchet derivative of $L(u,p)$ w.r.t to u in the direction v and w.r.t p in the direction q. We obtain: find $(u,p) \in V \times Q$ such that

$$\int_\Omega \nabla u : \nabla v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x - \int_\Omega p \cdot \nabla \times v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x,$$

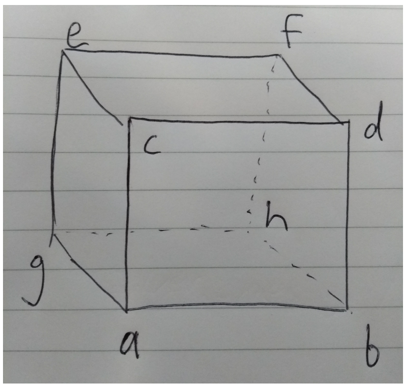
$$b(u,q) = -\int_\Omega q \cdot \nabla \times u \,\mathrm{d}x = 0,$$

for all $(v,q) \in V \times Q$.

We consider the Lagrange finite element $[CG_1]^3 \times [CG_1]^3$, i.e. piecewise linear basis functions. Let V_h and Q_h be the finite element function space that arises from equipping each cell $\mathcal{K}$ of a mesh $\mathcal{M}$ with this element.

We then do the same as in the notes by constructing a spurious pressure mode $p_h \neq 0 \in Q_h$ such that $b(v_h, p_h) = 0$ for all $v_h \in V_h$. This implies the problem does not satisfy the inf-sup condition for saddle point problems (14.7.10) in the notes.

The students can now come up with a discretized domain such that there is a spurious mode. For example, Let Ω be a cube, and divide this cube into six tetrahedrons such that the functions defined below are zero. We label the nodes as in the following figure.



The six tetrahedrons have nodes abch, bcdh, chdf, agch, gech, ecfh. Since it's P1, p_h is a pressure vector field determined completely by its degrees of freedom at the vertices, thus we specify its values there: let

$$p_h^{(l)}(i,j) = \begin{cases} 1 & \text{on nodes } a, b, d, e, f, g, \\ -1 & \text{on nodes } c, h, \end{cases}$$

where $p_h^{(l)}$, $l = 1, 2, 3$ are the components of p_h.

These functions are not equal to zero but have integral zero (tetrahedron centroid quadrature is exact for P1). Now, since v_h is piecewise linear on each tetrahedron $\mathcal{K}$, each component of $(\nabla \times v_h)|_{\mathcal{K}}$, $(\nabla \times v_h)^{(l)}|_{\mathcal{K}}$ is a constant, $l = 1, 2, 3$. Therefore, for arbitrary $v_h \in V_h$,

$$\begin{aligned} b(v_h, p_h) &= -\int_\Omega p_h \cdot \nabla \times v_h \,\mathrm{d}x \\ &= -\int_\Omega \left(p_h^{(1)} (\nabla \times v_h)^{(1)} + p_h^{(2)} (\nabla \times v_h)^{(2)} + p_h^{(3)} (\nabla \times v_h)^{(3)} \right) \mathrm{d}x \\ &= -\sum_{\mathcal{K} \in \mathcal{M}} \left[(\nabla \times v_h)^{(1)}|_{\mathcal{K}} \int_{\mathcal{K}} p_h^{(1)} \,\mathrm{d}x + (\nabla \times v_h)^{(2)}|_{\mathcal{K}} \int_{\mathcal{K}} p_h^{(2)} \,\mathrm{d}x + (\nabla \times v_h)^{(3)}|_{\mathcal{K}} \int_{\mathcal{K}} p_h^{(3)} \,\mathrm{d}x \right] \\ &= 0. \end{aligned}$$

Therefore, the discrete inf-sup condition:

$$0 < \gamma \le \inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q},$$

cannot be satisfied.

7. (Advanced, optional)

Solution: Assume the induction hypothesis $Zu_k = u_k$. First, examine the N-K iteration:

$$\begin{aligned} u_{k+1} &= u_k - F'(u_k)^{-1} F(u_k) \\ &= u_k - F'(u_k)^{-1} R^{-1} R F(u_k). \end{aligned}$$

Now consider Zu_{k+1}:

$$Zu_{k+1} = Zu_k - Z \left(F'(u_k)^{-1} R^{-1} \right) R F(u_k).$$

If Z commuted with $F'(u_k)^{-1} R^{-1}$, then we'd be happy. Assume this for now, and let's proceed:

$$\begin{aligned} Zu_{k+1} &= Zu_k - \left(F'(u_k)^{-1} R^{-1} \right) Z R F(u_k) \\ &= u_k - \left(F'(u_k)^{-1} R^{-1} \right) R F(Z u_k) \\ &= u_k - \left(F'(u_k)^{-1} R^{-1} \right) R F(u_k) \\ &= u_{k+1}. \end{aligned}$$

It remains to investigate whether Z does indeed commute with this operator. To see this, differentiate both sides of the symmetry relationship to yield

$$Z R F'(u; v) = R F'(Zu; Zv)$$

for all $u, v \in V$, and so

$$Z R F'(u) = R F'(Zu) Z.$$

Since Z is invertible, this implies

$$RF'(u) = Z^{-1}RF'(Zu)Z$$

and hence

$$\big(RF'(u)\big)^{-1} = Z^{-1}F'(Zu)^{-1}R^{-1}Z,$$

or in other words

$$ZF'(u)^{-1}R^{-1} = F'(Zu)^{-1}R^{-1}Z.$$

So the operators do commute, so long as $u = Zu$!