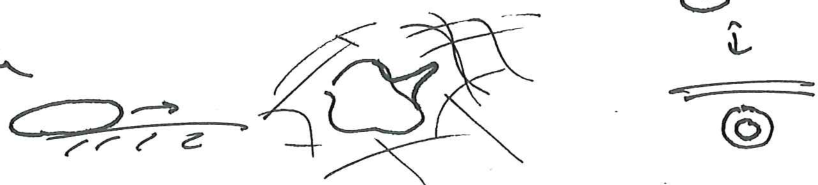
Biomembranes Overview

- Membrane : structure w/ one small dimension

Motivation ▫ Micro · the diverse and fascinating behaviour of cells highly linked to mechanics of cell membrane

Exs
- Permeability
- Exo-, endocytosis

- Cell migration



☆ Large shape change ☆

▫ Macro - a sheet of cells → tissue mechanics, eg skin

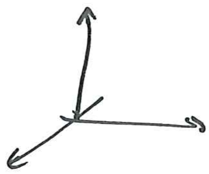
- In plants: leaves, petals ← Q: How do get shape?
  - Venus fly trap, some rapid seed dispersal triggered by release of elastic energy due to change in geom.

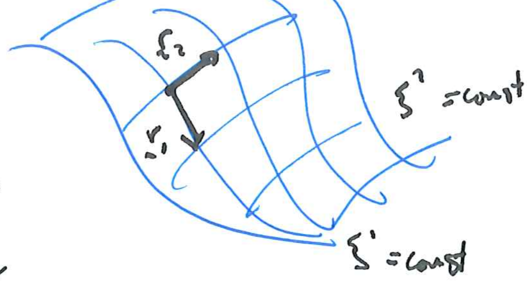
Plan Geometry of surfaces → Membrane Energy → Energy min.

- Alt : Force, moment balance (for axisymmetric)

Biomembranes

Geometry of surfaces





Let Σ be an orientable surface

w/ parameterisation $\underset{\sim}{x}(\xi^1, \xi^2) \in \mathbb{R}^3$, $(\xi^1, \xi^2) \in M \subset \mathbb{R}^2$

- we assume $\underset{\sim}{x}$ is at least C^2 & such that

$$\underset{\sim}{r}_i := \frac{\partial \underset{\sim}{x}}{\partial \xi^i} \quad \text{are lin. indep} \quad \forall \, (\xi^1, \xi^2) \in M.$$

Can define a normal $\underset{\sim}{n} = \dfrac{\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2}{\|\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2\|}$

and $\{\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{n}\}$ forms a basis

Surface area $\quad A = \displaystyle\int_\Sigma dS$

Recall (1^{st} yr) $\quad d\underset{\sim}{S} = \underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2 \, d\xi^1 d\xi^2$, $\quad dS = |d\underset{\sim}{S}|$.

Using identity $(\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2)^2 = \underset{\sim}{r}_1^2 \underset{\sim}{r}_2^2 - (\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2)^2$, we have

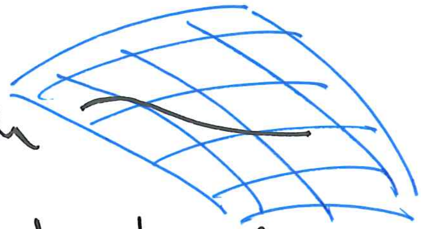
$$dS = \sqrt{\underset{\sim}{r}_1^2 \underset{\sim}{r}_2^2 - (\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2)^2} \, d\xi^1 d\xi^2$$

Def'n Let $g_{ij} := \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j = \dfrac{\partial \underset{\sim}{x}}{\partial \xi^i} \cdot \dfrac{\partial \underset{\sim}{x}}{\partial \xi^j}$. This is the metric tensor. Also define $G = (g_{ij})$ - the matrix of metric tensor.

Then $dS = \sqrt{g_{11}g_{22} - g_{12}^2}\; d\xi^1 d\xi^2 = \sqrt{\det G}\; d\xi^1 d\xi^2$

and $A = \int_M \sqrt{\det G}\; d\xi^1 d\xi^2$

<u>Arclength</u> - given a path



on Σ, the infinitesimal length

$$ds^2 = \left| \underset{\sim}{x}(\xi^1 + d\xi^1, \xi^2 + d\xi^2) - \underset{\sim}{x}(\xi^1, \xi^2) \right|^2$$

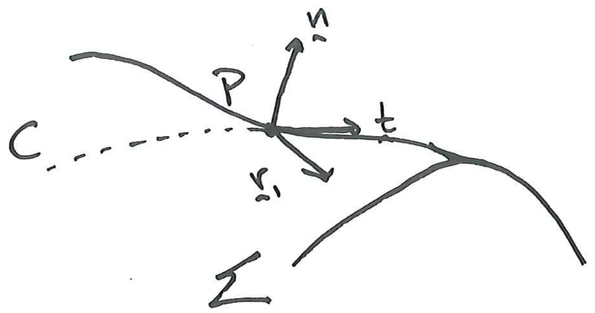
$$\overset{\text{(expand)}}{=} \left| \underset{\sim}{r}_1\, d\xi^1 + \underset{\sim}{r}_2\, d\xi^2 \right|^2 = \underset{\sim}{r}_1^2 (d\xi^1)^2 + \underset{\sim}{r}_2^2 (d\xi^2)^2 + 2\, \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2\, d\xi^1 d\xi^2$$

$$= \underline{g_{ij}\, d\xi^i d\xi^j} \quad \text{(w/ summation convention)}$$

↑ first fundamental form

$$\therefore\ L = \int_a^b \sqrt{g_{ij} \frac{d\xi^i}{dt} \frac{d\xi^j}{dt}}\; dt$$ is arclength of curve $\underset{\sim}{x}(\xi^1(t), \xi^2(t))$ for $t \in [a,b]$

Normal curvature



Consider curve C on Σ passing thru pt P and parameterised by arclength s

$\underset{\sim}{t}(s) = \frac{d\underset{\sim}{x}}{ds}$ is unit tangent

Let $\underset{\sim}{K} := \frac{d\underset{\sim}{t}}{ds}$. We know $\underbrace{|K|}_{\text{curvature of } C} = \left|\frac{d\underset{\sim}{t}}{ds}\right|$

Let $\underset{\sim}{K} = -K_n \underset{\sim}{n} + \underset{\sim}{K_g}$ w/ $\underset{\sim}{K_g} \cdot \underset{\sim}{n} = 0$

Normal curvature [sign so that sphere w/ outward normal has $K_n > 0$]

Due to fact surface is curved in $\mathbb{R}^3$

$K_g = |\underset{\sim}{K_g}|$ geodesic curv.

curve is curved relative to surface

"Consider following latitude vs longitude (geodesic) on a sphere"

$$\therefore \underset{\sim}{K} = \frac{d\underset{\sim}{t}}{ds} = \frac{d^2\underset{\sim}{x}}{ds^2} = \frac{d}{ds}\left(\frac{\partial \underset{\sim}{x}}{\partial \xi^i}\frac{d\xi^i}{ds}\right) = \frac{\partial^2 \underset{\sim}{x}}{\partial\xi^i \partial\xi^j}\frac{d\xi^j}{ds}\frac{d\xi^i}{ds} + \underbrace{\frac{d\underset{\sim}{x}}{d\xi^i}}_{= \underset{\sim}{r}_i}\frac{d^2\xi^i}{ds^2}$$

$$\text{so } K_n = -\underset{\sim}{n}\cdot\underset{\sim}{K} = \underbrace{-\underset{\sim}{n}\cdot\frac{\partial^2 \underset{\sim}{x}}{\partial\xi^i\partial\xi^j}}_{= K_{ij},\ \text{symmetric}}\frac{d\xi^j}{ds}\frac{d\xi^i}{ds} \quad \text{since } \underset{\sim}{n}\cdot\underset{\sim}{r}_i = 0$$

Define $K_{ij} = K_{ji} = -\underset{\sim}{n}\cdot\frac{\partial \underset{\sim}{r}_i}{\partial \xi^j}$. Then $K_n = K_{ij}\frac{d\xi^j}{ds}\frac{d\xi^i}{ds}$

extrinsic curv. tensor

- $K_{ij}\, d\xi^i d\xi^j$ is the 2nd fundamental form

Extremal values of k_n

$$K_{ij}\, d\xi^i d\xi^j = k_n\, ds^2 = k_n\, g_{ij}\, d\xi^i d\xi^j$$

$$\Rightarrow \left(K_{ij} - k_n g_{ij}\right) d\xi^i d\xi^j = 0 = \left(K_{ij} - k_n g_{ij}\right) \frac{d\xi^i}{ds} \frac{d\xi^j}{ds}$$

g_{ij}, K_{ij} depend on surface only (& not on curve C), while $dk_n = 0$ at an extremal of varying the paths of curves through P. $\therefore$ Differentiate wrt $\frac{d\xi^p}{ds}$ ☆

↑ like the dir. of the path

$$0 = \left(K_{ij} - k_n g_{ij}\right)\left(\delta^i_p \frac{d\xi^j}{ds} + \frac{d\xi^i}{ds} \delta^j_p\right)$$

↖ Kronecker delta, $=1$ if $i=p$

$$= \left(K_{pj} - k_n g_{pj}\right) \frac{d\xi^j}{ds} + \left(K_{ip} - k_n g_{ip}\right) \frac{d\xi^i}{ds}$$

Both terms same since K_{ij}, g_{ij} are symmetric

$$= 2\left(K_{pj} - k_n g_{pj}\right) \frac{d\xi^j}{ds}$$

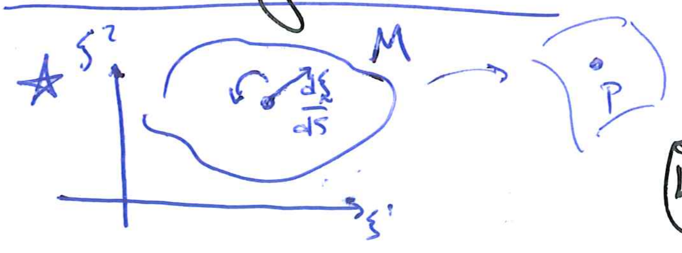
$$\therefore\ 2\left(g^{\beta p} K_{pj} - k_n g^{\beta p} g_{pj}\right) \frac{d\xi^j}{ds} = 0 \quad \text{w/} \quad g^{\beta p} g_{pj} = \delta^\beta_j$$

In matrix notation

$$\left(G^{-1} K - k_n I\right) \frac{d\underset{\sim}{\xi}}{ds} = \underset{\sim}{0}$$

$\therefore$ Extremal values of k_n are eigenvalues of

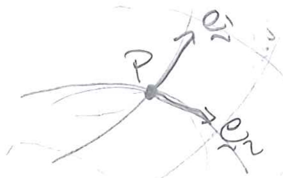
$$L = G^{-1} K$$



Let $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ denote the orthonormal eigenvectors of L w/ associated eigvalues k_1, k_2

principal curvatures

Can write $\underset{\sim}{t} = \cos\theta\, \underset{\sim}{e}_1 + \sin\theta\, \underset{\sim}{e}_2$



and (Euler, 1760) $k_n = k_1 \cos^2\theta + k_2 \sin^2\theta$

<u>Def'n</u> $H = \frac{\mathrm{Tr}(L)}{2} = \frac{k_1 + k_2}{2}$ mean curvature

$K_G = \det L = k_1 k_2$ $\underbrace{\text{Gaussian curvature}}_{\text{intrinsic to surface}}$

Note: $|H|$ is indep. of parameterisation, but H changes sign w/ dir. of normal vec

<u>Classification</u> $K_G > 0$ Elliptic, $K_G < 0$ Hyperbolic, $K_G = 0$ Parab.

<u>Gauss-Bonnet Thm</u> Let Σ be compact 2D surface w/ bd'y $\partial\Sigma$.

Then $\int_\Sigma K_G \, dS + \int_{\partial\Sigma} k_g \, ds = 2\pi \underline{\chi(\Sigma)}$

Euler characteristic, $= 2 - 2p$ for surface of genus p
[sphere - genus 0
Torus - genus 1, etc]

$\therefore$ For closed, orientable surface

$$\int_\Sigma K_G \, dS = 4\pi(1-p)$$

Fluid Biomembranes

· Lipid bilayer

Assumptions

- sufficiently thin so can be represented by surface, Σ
- No resistance to shear, but resists bending & stretching
- Free energy given by (Helfrich 1973)

$$\mathcal{E} = \int_{\Sigma} dS \left(\underbrace{\gamma}_{\text{surface tension}} + 2\underbrace{\kappa}_{\text{Bending modulus}} \left(H - \underbrace{H_0}_{\text{intrinsic mean curv.}}\right)^2 + \underbrace{\bar{\kappa}}_{\text{saddle-splay modulus}} K_G \right)$$

[Note $2\kappa H^2 = \frac{\kappa}{2}(k_1 + k_2)^2$] ⇒ if flat in one direction, $k_2 = 0$, $K_G = 0$
→ recover beam energy

Goal: find shape that minimises $\mathcal{E}$, given ref. shape

- if extra constraints, add Lagrange multiplier.

eg if fixed volume $V = V_0$, minimise

$$\mathcal{E} - \underbrace{P}_{\text{Lag. mult - pressure}}(V - V_0)$$

- For closed surface, Gauss-Bonnet ⇒ can ignore K_G

Estimates

$$[H] = \frac{1}{\text{Length}} \Rightarrow [\kappa] = \text{Energy}$$

$$[\gamma] = \frac{\text{Energy}}{\text{Length}^2} \Rightarrow \left(\frac{\kappa}{\gamma}\right)^{\frac{1}{2}} =: \lambda$$

is length - charact. length at which both bending & tension ~~$\frac{?}{}$~~ matter.

If L is lengthscale of variation in membrane,

$L \gg \lambda \Rightarrow$ surface tension dominant

$L \ll \lambda \Rightarrow$ bending dominant

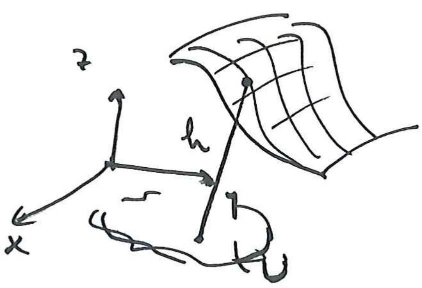
eg ~~for~~ lipid bilayers: $\kappa \simeq 10^{-19}\,J$, $\gamma \simeq 10^{-3}\,\frac{N}{m} = 10^{-3}\,\frac{J}{m^2}$

$\Rightarrow \lambda \simeq 10^{-8}\,m$

$\therefore$ For $L \gtrsim 10\,nm$, surface tension dominates

$\rightarrow$ will form approx. sphere

Monge Representation: we consider surface as fn $z = h(x,y) \in C^2$, $(x,y) \in U$



$\underset{\sim}{r} = (x, y, h(x,y))$

$\rightarrow \underset{\sim}{r}_1 = \frac{\partial \underset{\sim}{r}}{\partial x} = (1, 0, h_x)$, $\underset{\sim}{r}_2 = \frac{\partial \underset{\sim}{r}}{\partial y} = (0, 1, h_y)$

$$\underset{\sim}{n} = \underset{\text{by choice}}{-}\frac{\underset{\sim}{r}_1 \wedge \underset{\sim}{r}_2}{\|\cdot\|} = \frac{(+h_x, +h_y, -1)}{\sqrt{1 + h_x^2 + h_y^2}}$$

Recipe:
- Parameterisation $\rightarrow G, K \rightarrow L \rightarrow \kappa_1, \kappa_2 \rightarrow H, K_G \rightarrow \mathcal{E}$
- Euler-Lagrange $\rightarrow$ shape

Metric tensor matrix
$$G = \begin{pmatrix} \underset{\sim}{r}_1^2 & \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 \\ \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 & \underset{\sim}{r}_2^2 \end{pmatrix} = \begin{pmatrix} 1 + h_x^2 & h_x h_y \\ h_x h_y & 1 + h_y^2 \end{pmatrix}$$

$\rightarrow \det G = 1 + h_x^2 + h_y^2$, recall g

$$G^{-1} = \frac{1}{\det G}\begin{pmatrix} 1 + h_y^2 & -h_x h_y \\ -h_x h_y & 1 + h_x^2 \end{pmatrix}$$

Now consider $K = \left(K_{ij} \right)$, $K_{ij} = -\underset{\sim}{n} \cdot \dfrac{\partial \underset{\sim}{r}_i}{\partial x^j}$

$$K_{11} = \frac{(-h_x, -h_y, 1)}{\sqrt{g}} \cdot (0, 0, h_{xx}), \quad K_{12} = \frac{(-h_x, -h_y, 1)}{\sqrt{g}} \cdot (0, 0, h_{xy})$$

$$K_{22} = \quad " \quad " \quad \cdot (0, 0, h_{yy})$$

$$\Longrightarrow \quad K = \frac{1}{\sqrt{g}} \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}$$

Can now compute $L = G^{-1} K$, from which we get

$$K_G = \det L = \det G^{-1} \det K = \frac{\det K}{\det G} = \frac{h_{xx} h_{yy} - h_{xy}^2}{g^2}$$

$$H = \frac{\operatorname{tr}(L)}{2} = \frac{1}{2 g^{3/2}} \left(h_{xx}(1 + h_y^2) + h_{yy}(1 + h_x^2) - 2 h_{xy} h_x h_y \right)$$

Note: can also write as

$$2H = \nabla \cdot \left(\frac{\nabla h}{\sqrt{g}} \right) = -\nabla \cdot \underset{\sim}{n}$$

Simple case - Surface tension only

Set $\kappa = \bar{\kappa} = H_0 = 0$, γ const $\rightarrow$ $\mathcal{E} = \gamma \int_{\Sigma} dS$

(wants to minimise surface area)

$$\mathcal{E} = \gamma \int_{M} \sqrt{\det G}\, d\xi^1 d\xi^2 = \gamma \int \underbrace{\sqrt{1 + h_x^2 + h_y^2}}_{\mathcal{L}(h,\, h_x,\, h_y)}\, dx\, dy$$

Euler-Lagrange eq'n is

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial h_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial h_y}\right) - \frac{\partial \mathcal{L}}{\partial h} = 0$$

$$\rightarrow \quad \frac{\partial}{\partial x}\left(\frac{h_x}{\sqrt{1+h_x^2+h_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{h_y}{\sqrt{1+h_x^2+h_y^2}}\right) = 0$$

ie $\nabla \cdot \left(\dfrac{\nabla h}{\sqrt{g}}\right) = 0$ $\qquad \left(\nabla = \underset{\sim}{e}_x \dfrac{\partial}{\partial x} + \underset{\sim}{e}_y \dfrac{\partial}{\partial y}\right)$

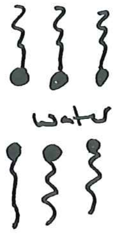
$\therefore$ $H = 0$ or equiv $\nabla \cdot \hat{\underset{\sim}{n}} = 0$

ie Surface w/ zero mean curv

A surface satisfying $H \equiv 0$ is called a <u>minimal surface</u> - locally minimises surface area at every point.

- Note $H = 0 \Rightarrow k_1 = -k_2 \Rightarrow K_G \leq 0$ (w/ equality only if $k_1 = k_2 = 0$, flat!)
  $\Rightarrow$ minimal surfaces are saddle shaped.

<u>Example</u> Soap film spanning any boundary



Bilayer w/ $L \gg \gamma$

- consider soap film b/t two rings

- Surface of revolution

$$\underset{\sim}{x} = \begin{pmatrix} f(z)\cos\theta \\ f(z)\sin\theta \\ z \end{pmatrix}$$

Then $\underset{\sim}{r}_1 = \dfrac{\partial \underset{\sim}{x}}{\partial z} = \begin{pmatrix} f'\cos\theta \\ f' s \\ 1 \end{pmatrix}$, $\underset{\sim}{r}_2 = \begin{pmatrix} -f\sin\theta \\ f\cos\theta \\ 0 \end{pmatrix} \rightarrow G = \begin{pmatrix} 1+f'^2 & 0 \\ 0 & f^2 \end{pmatrix}$

So $\mathcal{E} = \gamma \iint \sqrt{\det G}\, d\theta dz$ w/ $\sqrt{\det G} = f(z)\sqrt{1+f'^2} =: F$

Euler Lagrange $\rightarrow \dfrac{1+f'^2 - ff''}{(1+f'^2)^{3/2}} = 0$, same as $H = 0$

Better: Beltrami identity: $\dfrac{\partial F}{\partial z} = 0 \rightarrow F - f'\dfrac{\partial F}{\partial f'} = C$ const $\rightarrow$ 1st integral

$\dfrac{f}{\sqrt{1+f'^2}} = C$. can separate and integrate

Small gradient approx.

Keep $\bar{\kappa} = H_0 = 0$, but $\kappa \neq 0$ and supp. $|h_x|, |h_y| \ll 1$

Then $\sqrt{g} = (1 + h_x^2 + h_y^2)^{\frac{1}{2}} \approx 1 + \frac{1}{2}(h_x^2 + h_y^2) + O(\nabla h^3)$

$$\Rightarrow (2H)^2 = \frac{1}{g^{3}}\left(h_{xx} + h_{yy} + O(\nabla h^3)\right)^2 = (h_{xx} + h_{yy})^2 (1 + O(\nabla h^2))$$

$$\therefore \mathcal{E} = \int_{\Sigma} \cancel{dxdy}\, dS\,(\gamma + 2\kappa H^2) = \int_U dx\,dy\, \sqrt{g}\,(\gamma + 2\kappa H^2)$$

$$= \int_U dx\,dy \left(1 + \tfrac{1}{2}(\nabla h)^2\right)\left(\gamma + \tfrac{\kappa}{2}(\nabla^2 h)^2\right) + \dots$$ h.o.t.

$$= \int dx\,dy \left(\gamma + \tfrac{1}{2}\gamma(\nabla h)^2 + \tfrac{1}{2}\kappa(\nabla^2 h)^2\right) + \text{h.o.t.}$$

constant, won't impact minimisation.

Variation: $$\frac{d}{d\varepsilon}\mathcal{E}[h + \varepsilon\eta(x,y)]\Big|_{\varepsilon=0} = \int_U \gamma\, \underbrace{\nabla h \cdot \nabla\eta}_{= \nabla\cdot(\eta\nabla h) - \eta\nabla^2 h} + \kappa\, \underbrace{\nabla^2 h\, \nabla^2 \eta}_{= \nabla\cdot(\nabla\eta\,\nabla^2 h) - \nabla\eta\cdot\nabla\nabla^2 h}\, dx\,dy$$

$$= \int_{\partial U} (\gamma\eta\nabla h + \kappa\nabla\eta\,\nabla^2 h)\cdot \underset{\sim}{N}\, ds - \int_U \gamma\eta\nabla^2 h + \kappa\underbrace{\nabla\eta\cdot\nabla(\nabla^2 h)}_{= \nabla\cdot(\eta\nabla\nabla^2 h) - \eta\nabla^4 h}\, dx\,dy$$

$$= \int_{\partial U} \Big(\eta\underbrace{(\gamma\nabla h - \kappa\nabla\nabla^2 h)}_{(A)} + \kappa\nabla\eta\,\underbrace{\nabla^2 h}_{(B)}\Big)\cdot \underset{\sim}{N}\, ds - \int_U (\gamma\nabla^2 h + \kappa\nabla^4 h)\,\eta\, dx\,dy$$

For an extremum, we require

$$\boxed{\nabla^4 h - \frac{1}{\lambda^2}\nabla^2 h = 0}, \quad \lambda := \sqrt{\frac{\kappa}{\gamma}}$$

(Note: $\lambda \ll 1 \Rightarrow$ singular pert. problem!)

Bdy cond: Need (A) & (B) to vanish also.

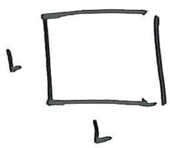
- If impose h on $\partial\Sigma$ $\Rightarrow$ must have $\eta = 0$ on $\partial\Sigma$ so (A) $= 0$, else require $\left(\nabla h - \frac{1}{\lambda^2}\nabla(\nabla^2 h)\right)\cdot \underset{\sim}{N} = 0$

- If impose $\nabla h \cdot \underset{\sim}{N}$ on $\partial\Sigma$ $\Rightarrow$ must have $\nabla\eta\cdot\underset{\sim}{N} = 0$ so (B) $= 0$, else require $\nabla^2 h = 0$ on $\partial\Sigma$.

Ex Flicker Spectroscopy

Study biomembrane mechanics by measuring spectrum of thermal undulations via light microscopy

- Small gradients ⇒ Linear eq'ns, so can use Fourier methods
- Consider square membrane (closed surfaces analysed in practice)



Seek $h(x,y) = h(\underset{\sim}{r}) = \sum_{\underset{\sim}{q}} h_{\underset{\sim}{q}} e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}$

where $\underset{\sim}{q} = \frac{2\pi}{L}(n_x, n_y)$, $n_x, n_y \in \mathbb{Z}$ Fourier modes

and $h_{\underset{\sim}{q}} \in \mathbb{C}$ are Fourier coeff. h real $\to h_{-\underset{\sim}{q}} = \overline{h_{\underset{\sim}{q}}}$

Then $\nabla h = \sum i\underset{\sim}{q}\, h_{\underset{\sim}{q}} e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} \Rightarrow (\nabla h)^2 = \sum_{\underset{\sim}{q},\underset{\sim}{q}'} -\underset{\sim}{q}\cdot\underset{\sim}{q}'\, h_{\underset{\sim}{q}} h_{\underset{\sim}{q}'} e^{i\underset{\sim}{r}\cdot(\underset{\sim}{q}+\underset{\sim}{q}')}$

$\nabla^2 h = \sum -\underset{\sim}{q}^2 h_{\underset{\sim}{q}} e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} \Rightarrow (\nabla^2 h)^2 = \sum_{\underset{\sim}{q},\underset{\sim}{q}'} \underset{\sim}{q}^2 \underset{\sim}{q}'^2 h_{\underset{\sim}{q}} h_{\underset{\sim}{q}'} e^{i\underset{\sim}{r}\cdot(\underset{\sim}{q}+\underset{\sim}{q}')}$

$$\therefore \mathcal{E} = \frac{1}{2}\int dx\,dy \sum h_{\underset{\sim}{q}} h_{\underset{\sim}{q}'} e^{i\underset{\sim}{r}\cdot(\underset{\sim}{q}+\underset{\sim}{q}')}\left(\kappa\, \underset{\sim}{q}^2 \underset{\sim}{q}'^2 - \gamma\, \underset{\sim}{q}\cdot\underset{\sim}{q}'\right)$$

Orthogonality $\to \int dx\,dy\, e^{i\underset{\sim}{r}\cdot(\underset{\sim}{q}+\underset{\sim}{q}')} = L^2 \delta_{\underset{\sim}{0},\underset{\sim}{q}+\underset{\sim}{q}'}$

$$\Rightarrow \mathcal{E} = \frac{L^2}{2}\sum_{\underset{\sim}{q}} h_{\underset{\sim}{q}} h_{-\underset{\sim}{q}}\left(\kappa \underset{\sim}{q}^4 - \gamma \underset{\sim}{q}^2\right) = \frac{L^2}{2}\sum_{\underset{\sim}{q}} |h_{\underset{\sim}{q}}|^2 \left(\kappa \underset{\sim}{q}^4 - \gamma \underset{\sim}{q}^2\right)$$

$h_{-\underset{\sim}{q}} = \overline{h_{\underset{\sim}{q}}}$

and $\underset{\sim}{q}^2 := \underset{\sim}{q}\cdot\underset{\sim}{q}$

Goal : compute $\langle |h_p|^2 \rangle$ for a given mode
$p = \frac{2\pi}{L}(p_x, p_y)$

$$\langle |h_p|^2 \rangle = \frac{1}{Z} \int \underset{\substack{\uparrow \\ \text{space of} \\ \text{all } h}}{D[h]}\, |h_p|^2 e^{-\beta \varepsilon}$$

h_p is complex number, so treat as polar coords w/ $r = |h_p|$, $\theta = \arg(h_p)$

$$= \frac{1}{Z} \int \prod_m \underbrace{d|h_m|\, d\theta_m\, |h_m|}_{\text{"}r\,dr\,d\theta\text{"}} \left(|h_p|^2 e^{-\beta \frac{L^2}{2} \sum_q |h_q|^2 f(q)} \right)$$

$f(q) := \kappa q^4 - \gamma q^2$

and $$Z = \int \prod_m d|h_m|\, d\theta_m\, |h_m|\, e^{-\beta \frac{L^2}{2} \sum_q |h_q|^2 f(q)}$$

Key: Z has all the same integrals except at $m=p$
so everything cancels but

$$\langle |h_p|^2 \rangle = \frac{\int d|h_p|\, |h_p|^3 e^{-\beta \frac{L^2}{2} |h_p|^2 f(p)}}{\int d|h_p|\, |h_p|\, e^{-\beta \frac{L^2}{2} |h_p|^2 f(p)}}$$

$$= \frac{-1}{\frac{L^2}{2} f(p)} \frac{\frac{\partial}{\partial \beta} I_p}{I_p} \quad \text{w/} \quad I_p = \int d|h_p|\, |h_p|\, e^{-\beta \frac{L^2}{2} |h_p|^2 f(p)}$$

$$= \frac{-1}{\frac{L^2}{2} f(p)} \underbrace{\frac{\partial}{\partial \beta} \ln I_p} \quad = \frac{\partial}{\partial \beta} \ln\left(\frac{\text{const}}{\beta}\right) = -\frac{1}{\beta}$$

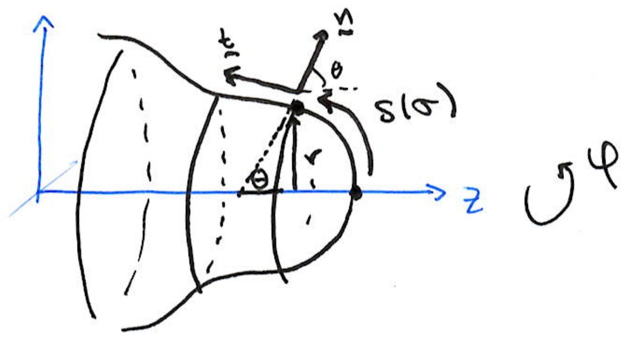
$$= \frac{k_b T}{\frac{L^2}{2} f(p)}$$

$$\therefore \boxed{\langle |h_p|^2 \rangle = \frac{2 k_b T}{L^2 (\kappa p^4 - \gamma p^2)}}$$

Axisymmetric membranes & Shells

1. Kinematics

We take the membrane to be a surface of revol'n w/:



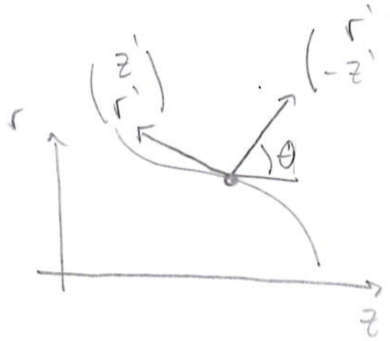
$\underset{\sim}{n}$ · unit normal

$\underset{\sim}{t} \equiv \underset{\sim}{e}_s$ unit tangent in direction of increasing arclength s

σ · material param, arclength <u>before</u> deformation

θ · angle b/t $\underset{\sim}{n}$ and axis of symmetry $\underset{\sim}{e}_z$

r · radius, dist. from $\underset{\sim}{e}_z$

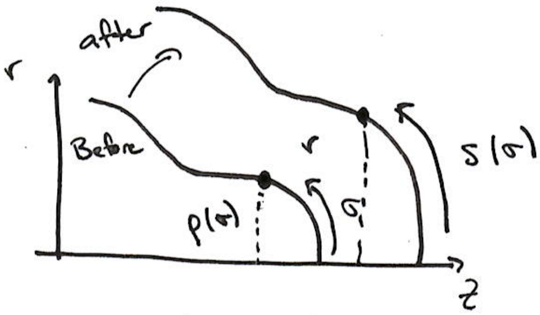


Trigonometry gives

$$\frac{dr}{ds} = \cos\theta, \quad \frac{dz}{ds} = -\sin\theta$$

Principal curvatures are $\kappa_s = \frac{d\theta}{ds}$, $\kappa_\varphi = \frac{\sin\theta}{r}$

Stretch variables



→ Stretch in s direction $\lambda_s = \frac{\partial s}{\partial \sigma}$

And radial stretch $\lambda_\varphi = \frac{r}{\rho}$

2. Mechanics

Consider an element of the membrane associated w/ section $[s, s+\Delta s]$, $[\varphi, \varphi+\Delta\varphi]$ subject to

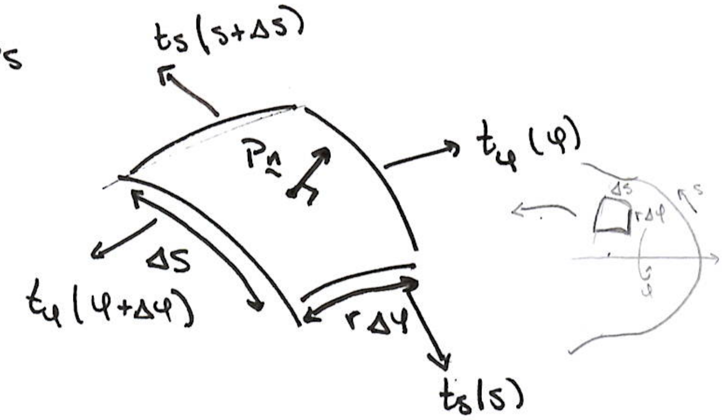
i) Pressure, $P\underset{\sim}{n}$, eg due to pressurised internal fluid

ii) Tension on surface $\underset{\sim}{T} = t_s \underset{\sim}{e}_s + t_\varphi \underset{\sim}{e}_\varphi$

unit vec. in direction of increasing azimuthal angle φ

iii) A force per unit area, eg due to external fluid movement. For axisymmetric this must take form $f \underset{\sim}{e}_s$

Zoom in on surface element :



Force balance, neglecting inertia (so $\underset{\sim}{a} = \underset{\sim}{0}$ in $\underset{\sim}{F} = m\underset{\sim}{a}$)

$$\rightarrow \Delta s \Delta\varphi \left[\underbrace{\frac{\partial}{\partial s}\left(r t_s \underset{\sim}{e}_s\right)} + \frac{\partial}{\partial\varphi}\left(t_\varphi \underset{\sim}{e}_\varphi\right) + rP\underset{\sim}{n} + rf\underset{\sim}{e}_s \right] = 0$$

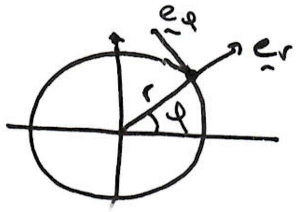
(neglecting higher order terms)

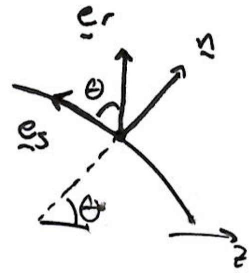
[eg. you'd have $r(s+\Delta s)\, t_s(s+\Delta s)\, \underset{\sim}{e}_s(s+\Delta s)\, \Delta\varphi$

$- r(s)\, t_s(s)\, \underset{\sim}{e}_s(s)\, \Delta\varphi$]

Now, we use $\dfrac{\partial t_\varphi}{d\varphi} = 0$

$$\frac{d \underset{\sim}{e}_\varphi}{d\varphi} = - \underset{\sim}{e}_r$$





$$= - \left(\cos\theta \, \underset{\sim}{e}_s + \sin\theta \, \underset{\sim}{n} \right)$$

Also, $\dfrac{\partial \underset{\sim}{e}_s}{\partial s} = \dfrac{\partial \underset{\sim}{t}}{\partial s} = - \kappa_s \underset{\sim}{n}$ by defn of curvature,

using $\underset{\sim}{t} = (-\sin\theta, \cos\theta)$

$\underset{\sim}{n} = (\cos\theta, \sin\theta)$ in

z–r coords.

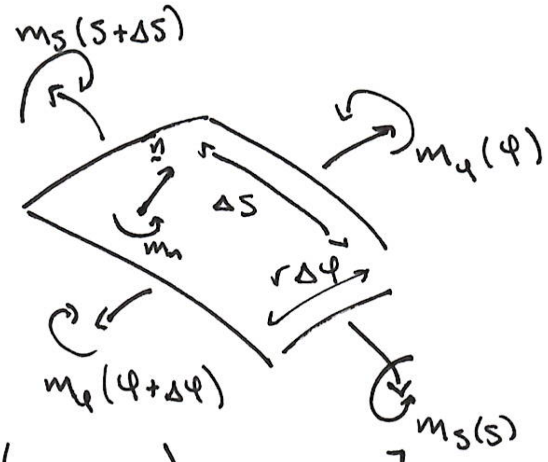
∴ Force balance becomes

$$\underset{\sim}{0} = \underset{\sim}{e}_s \left[\frac{\partial}{\partial s}(r t_s) - t_\varphi \cos\theta + r f \right] + \underset{\sim}{n} \left[- r t_s \kappa_s - t_\varphi \sin\theta + r P \right]$$

$$\therefore \boxed{\begin{aligned} P &= t_s \kappa_s + t_\varphi \frac{\sin\theta}{r} = t_s \kappa_s + t_\varphi \kappa_\varphi \\ \frac{\partial}{\partial s}(r t_s) &= t_\varphi \cos\theta - r f = t_\varphi \frac{\partial r}{\partial s} - r f \end{aligned}}$$

Note: if $t_s = t_\varphi$, get $P = 2tH$

Moment balance



~~Similar argument~~

$$\Delta s \Delta\varphi \left[\frac{\partial}{\partial s}(r m_s \underset{\sim}{e}_s) + \frac{\partial}{\partial \varphi}(m_\varphi \underset{\sim}{e}_\varphi) + m_n \underset{\sim}{n} \right] = \underset{\sim}{0}$$

Again, use $\frac{\partial \underset{\sim}{e}_\varphi}{\partial \varphi} = -\cos\theta\, \underset{\sim}{e}_s - \sin\theta\, \underset{\sim}{n}$, $\cancel{\frac{\partial m_\varphi}{\partial \varphi} = 0}$

$$\rightarrow \boxed{\frac{\partial}{\partial s}(r m_s) - m_\varphi \cos\theta = 0}$$

$\underset{\sim}{e}_\varphi$ direction: $\frac{\partial m_\varphi}{\partial \varphi} = 0$

$\underset{\sim}{n}$ direction gives m_n ito m_s, m_φ - <u>not</u> needed as m_n not imposed

Constant Pressure case $(f=0)$

Claim P const $\Rightarrow$ $r^2(2t_s K_\varphi - P) = \text{const}$

Pf System is: (i) $P = t_s K_s + t_\varphi K_\varphi = t_s \theta' + t_\varphi \frac{\sin\theta}{r}$ $\left(' = \frac{\partial}{\partial s}\right)$

(ii) $(r t_s)' = t_\varphi r'$, (iii) $r' = \cos\theta$

$$\cancel{(i) \Rightarrow (rP)' = r'P = (r t_s \theta')' + (t_\varphi \sin\theta)'}$$

$$(r K_\varphi)' = (\sin\theta)' = \cos\theta \cdot \theta' = \underline{r' K_s}$$

$$\therefore \quad P r' = \underline{r' K_s t_s} + \underbrace{r' K_\varphi t_\varphi}_{\text{use (ii)}} = (r K_\varphi)' t_s + (r t_s)' K_\varphi$$

$$\Rightarrow P r r' = r t_s (r K_\varphi)' + r K_\varphi (r t_s)' = (r^2 t_s K_\varphi)'$$

$$\Rightarrow \frac{P r^2}{2} - r^2 t_s K_\varphi = \text{const} \quad ✓$$

Observe: if surface crosses z-axis, so $r = 0$ at a pt,

then const $= 0 \Rightarrow P = 2 t_s K_\varphi$ (Young-Laplace Law!)

[Derivation of the const in Alain's book p 213]
↳ could give as additional material

<u>Constitutive Laws</u> - We need to relate t_s and t_φ to the deformation of the membrane, defined by the stretch ratios $\lambda_s = \frac{\partial s}{\partial \sigma}$, $\lambda_\varphi = \frac{r}{\rho}$

[there is a 3rd stretch, in normal direction, λ_3. For incompressible material (volume doesn't change),

$$\lambda_3 = \frac{1}{\lambda_s \lambda_\varphi} \quad]$$

General form: $t_s = A\, f_s(\lambda_s, \lambda_\varphi)$, $t_\varphi = A\, f_\varphi(\lambda_s, \lambda_\varphi)$

* such that $f_s(1,1) = f_\varphi(1,1) = 0$

<u>Standard moment constit relation</u>

$$m_s = m_\varphi = B\left(\kappa_s + \kappa_\varphi - K_0\right)$$

isotropic

sum of curvatures in stress-free config

For this, mom. balance becomes

$$\frac{\partial}{\partial s}(r m_s) - \underset{m_s}{\underbrace{m_\varphi}} \underset{\frac{\partial r}{\partial s}}{\underbrace{\cos\theta}} = 0 \Rightarrow \frac{\partial m_s}{\partial s} = 0$$

$$\Rightarrow \frac{\partial \kappa_s}{\partial s} + \frac{\partial \kappa_\varphi}{\partial s} = 0 \quad (\text{if } K_0 \text{ const})$$

$$\Rightarrow \kappa_s + \kappa_\varphi = K_1 \text{ , constant}$$

$$\Rightarrow \kappa_s = K_1 - \frac{\sin\theta}{r}$$

Summary - For const pressure, the shape $\begin{pmatrix} r(\sigma) \\ z(\sigma) \end{pmatrix}$ det'd by:

$$\left\{ \begin{array}{ll} \dfrac{ds}{d\sigma} = \lambda_s & (1) \\[2ex] \dfrac{dz}{d\sigma} = -\lambda_s \sin\theta & (2) \\[2ex] \dfrac{dr}{d\sigma} = \lambda_s \cos\theta & (3) \\[2ex] \dfrac{d\theta}{d\sigma} = \lambda_s K_s = \lambda_s \left(K_1 - \dfrac{\sin\theta}{r} \right) & (4) \\[2ex] \dfrac{dt_s}{d\sigma} = \lambda_s A \dfrac{\cos\theta}{r} \left(f_\varphi - f_s \right) & (5) \end{array} \right.$$

(1)–(2): Geometry

(4): Geometry + Mom. balance

(5): ← Force bal. + Constit.

FB is

$$\left[r' t_s + r t_s' = r' t_\varphi \quad \text{w/} \quad r' = \cos\theta \right]$$

and $P = t_s K_s + t_\varphi K_\varphi \Rightarrow P = A f_s \left(K_1 - \frac{\sin\theta}{r} \right) + A f_\varphi \cdot \frac{\sin\theta}{r}$ (6)

Notes

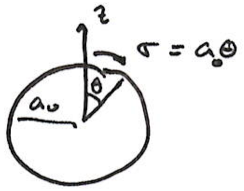
- Forms a system of 5 ODEs and 1 extra for 6 variables $\{s, z, r, \theta, t_s, \lambda_s\}$
- $f_s(\lambda_s, \frac{r}{\rho})$, $f_\varphi(\lambda_s, \frac{r}{\rho})$ would be given
- A typical problem: impose P + Bd'y cond, determine deformed shape
- $r = \rho(\sigma)$, $z = z_0(\sigma)$ (unstressed shape) needed to determine K_1
- Common to have wall softening/stiffening via $A = A(\sigma)$

Example - Inflation of a Sphere

Consider a sphere of radius a_0 in undeformed state. Supposing the pressure across the surface is increased to P^*, we wish to find the deformed radius, a.

Initial

$(\lambda_s, \lambda_\varphi = 1)$



$P = 0$

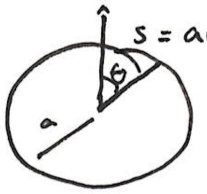
Geometry: $\left({}' = \frac{d}{d\sigma}\right)$ $\quad z_0' = -\sin\theta = -\sin\left(\frac{\sigma}{a_0}\right)$

$\rho' = \cos\theta$

w/ $z_0(0) = a_0 \rightarrow z_0 = a_0 \cos\left(\frac{\sigma}{a_0}\right)$

& $\rho(0) = 0 \rightarrow \rho = a_0 \sin\left(\frac{\sigma}{a_0}\right)$

Deformed



$P = P^* > 0$

$s = a\theta = \frac{a}{a_0}\sigma \Rightarrow \lambda_s = \frac{ds}{d\sigma} = \frac{a}{a_0}$

$z(\sigma) = a\cos\left(\frac{\sigma}{a_0}\right)$ satisfies $z' = -\lambda_s \sin\theta$

& $r(\sigma) = a\sin\left(\frac{\sigma}{a_0}\right)$

- Mom. balance: (4): $\theta' = \frac{1}{a_0} = \lambda_s\left(\kappa_1 - \frac{\sin\theta}{r}\right) = \frac{a}{a_0}\left(\kappa_1 - \frac{1}{a}\right)$

  satisfied w/ $\kappa_1 = \frac{2}{a}$ (constant as req'd)

- Constit: supp. $t_s = t_\varphi = A\left(\lambda_s^2 + \lambda_\varphi^2 - 2\right) \leftarrow f_s = f_\varphi$ by symmetry

  $\lambda_s = \frac{a}{a_0}$, $\lambda_\varphi = \frac{r}{\rho} = \frac{a}{a_0} \Rightarrow t_s = 2A\left(\frac{a^2}{a_0^2} - 1\right)$ is const

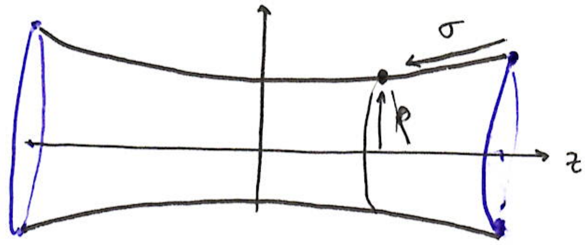
  so (5) is satisfied

- Normal Force bal: $\kappa_s = \frac{d\theta}{ds} = \frac{1}{a}$, $\kappa_\varphi = \frac{\sin\theta}{r} = \frac{1}{a}$, so $P = t_s\kappa_s + t_\varphi\kappa_\varphi$

$\rightarrow P^* = \frac{4A}{a}\left(\frac{a^2}{a_0^2} - 1\right) \Rightarrow a = \frac{a_0}{2}\left(\beta + \sqrt{\beta^2 + 4}\right)$

$\beta := \frac{P^* a_0}{4A}$

More challenging example - inflation of catenoid



Membrane pinned on two rings, radius b, at $z = \pm a$

1. Ref shape : Need to solve
$$\begin{cases} z_0' = -\sin\theta \\ \rho' = \cos\theta \\ \theta_0' = K_1 - \dfrac{\sin\theta}{\rho} \end{cases} \qquad \left(' = \frac{d}{d\sigma} \right)$$

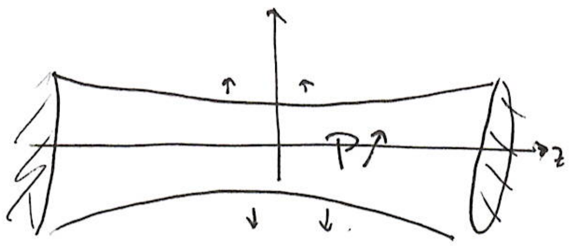
$K_1 = K_s + K_\varphi$ unknown apriori

subj. to $z_0(0) = a$ $\quad z_0(L) = -a$

$\rho(0) = b$ $\quad \rho(L) = b$

(given a, b, L)

2. Now inflate - increase P



$$s' = \lambda_s$$
$$z' = -\lambda_s \sin\theta$$
$$r' = \lambda_s \cos\theta$$
$$\theta' = \lambda_s \left(K_1 - \frac{\sin\theta}{r} \right)$$
$$t_s' = \lambda_s A \cdot \frac{\cos\theta}{r} \left(f_\varphi - f_s \right)$$
$$P = A \cdot f_s \cdot \left(K_1 - \frac{\sin\theta}{r} \right) + A \cdot f_\varphi \cdot \frac{\sin\theta}{r}$$

BC

$s(0) = 0$, $z(0) = a$, $r(0) = b$

$z(L) = -a$, $r(L) = b$

Highly non-linear!

Constit e.g. neo-Hookean (simplest)

$$f_s = \mu \left(\lambda_s^2 - \frac{1}{\lambda_s^2 \lambda_\rho^2} \right) \quad \text{w/} \quad \lambda_\varphi = \frac{r(\sigma)}{\rho(\sigma)}$$
$$f_\rho = \mu \left(\lambda_\varphi^2 - \frac{1}{\lambda_s^2 \lambda_\rho^2} \right)$$