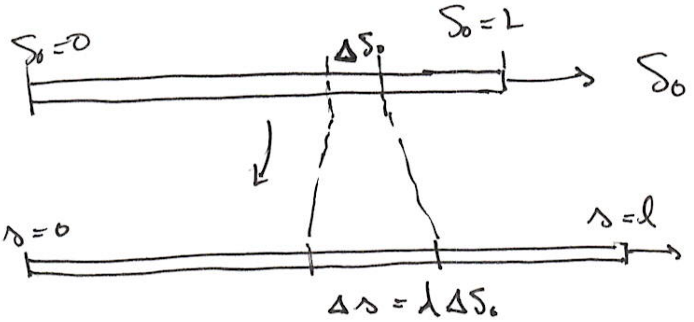
BIO GROWTH

1D Growth — We consider a 1D rod constrained to a line that deforms due to growth (increase of mass) and/or elastic response (stretching / compression)



S_0 — initial arclength

s — current arclength

$$\frac{\partial s}{\partial S_0} =: \lambda = \lambda(S_0) \text{ is stretch} \qquad \left(\lambda > 0 \Rightarrow \text{1-1 map} \right)$$

Purely elastic deformation (no growth) $\quad \lambda \overset{\text{all}}{=} \alpha$

Let σ be axial stress in rod $\left(\text{force} / \underset{= A}{\text{area section}} \right)$

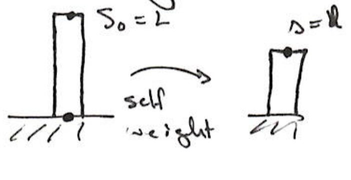
Define $\quad n = \frac{\sigma}{A} \qquad (= n_3$ from BioFilaments $)$

Force balance: $\quad \frac{dn}{dS_0} + f = 0$, $\quad f$: body force per unit (initial) length

Constit. law: $\quad n = h(\alpha) \quad$ w/ $\quad h(1) = 0$

eg Hooke's law $\quad n = EA(\alpha - 1)$

Ex



"Find l"

$f = -\rho g$, $\quad n(L) = 0 \;\rightarrow\; n = \rho g (S_0 - L) \;\rightarrow\; \alpha = \frac{\rho g}{EA}(S_0 - L) + 1$

$\rightarrow l = s(L) = \int_0^L \alpha \, dS_0 = L - \frac{\rho g L^2}{2EA}$

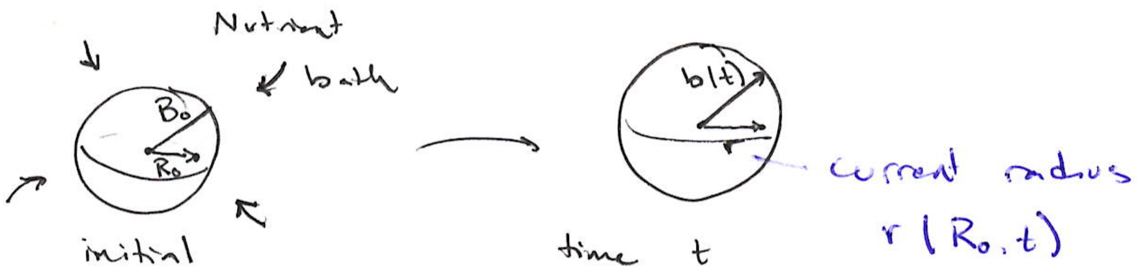
Pure Growth def. (No elasticity)

$\lambda \overset{\text{call}}{=} \gamma$ - Growth process : $\gamma = \gamma(t)$ follows a growth law $\dfrac{\partial \gamma}{\partial t} = G(\gamma, s, S_0, \ldots)$

e.g. $\dfrac{\partial \gamma}{\partial t} = k\gamma \rightarrow \gamma = e^{kt} \quad (\gamma(S_0, 0) = 1)$

& $\gamma = \dfrac{\partial s}{\partial S_0} \Rightarrow s = S_0 e^{kt}$

Application : tumour spheroid



Assume : i) Isotropic growth (same in all directions), expon. in time, proportional to nutrient concent. $u(r,t)$

ii) Nutrient diffuses in from bath

iii) Constant nutrient conc. at outer surface

Define volumetric growth by $dv = \eta \, dV_0$

For sphere, $dv = r^2 \sin\varphi \, dr \, d\theta \, d\varphi$, $dV_0 = R_0^2 \sin\varphi \, dR_0 \, d\theta \, d\varphi$

$$\Rightarrow r^2 dr = \eta R_0^2 dR_0 \Rightarrow \frac{\partial r}{\partial R_0} = \underbrace{\eta R_0^2 r^{-2}}_{= \gamma}$$

Growth : $\qquad \dfrac{\partial \eta}{\partial t} = k \eta u(r,t)$ $\qquad$ Discuss why ∇^2 is in r, not R_0

Nutrient : $\qquad \dfrac{\partial u}{\partial t} = D \nabla^2 u - \underset{\sim}{Q} = \dfrac{D}{r^2} \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r} \right) - Q$

diffusion const. $\qquad$ uptake (constant)

w/ $\quad u(b(t), t) = u_b$ (const)

- "Fast diffusion" : $u_t \approx 0 \rightarrow$ can solve for

$u = u(r)$ separately, then plug into

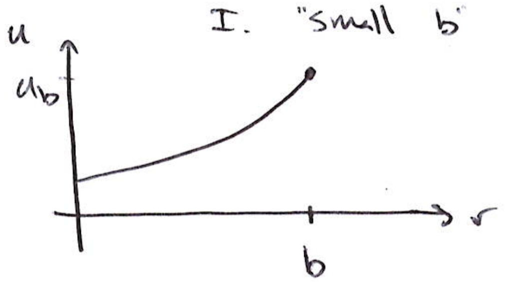
$$\frac{\partial \eta}{\partial t} = k \eta u, \qquad \frac{\partial r}{\partial R_0} = \frac{\eta R_0^2}{r^2}$$

We have $\dfrac{d}{dr}\left(r^2 \dfrac{du}{dr} \right) = \dfrac{Q}{D} r^2 \Rightarrow \dfrac{du}{dr} = \dfrac{Qr}{3D} + \dfrac{c_1}{r^2}$

~~Int. twice~~ $\Rightarrow \quad u = \dfrac{Qr^2}{6D} - \dfrac{c_1}{r} + c_2$

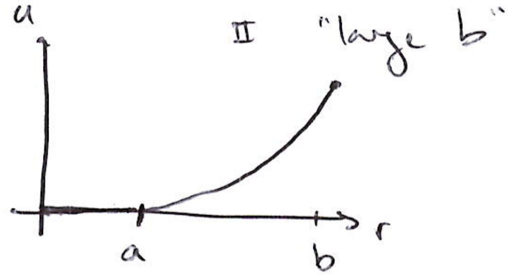
<u>Key</u> Must have $u \geq 0 \Rightarrow$ can either have

I. "small b"



OR

II "large b"



Case I : Set $c_1 = 0$, find c_2 via $u(b) = u_b$

$$\rightarrow \quad u = \frac{Q}{6D}\left(r^2 - b^2 \right) + u_b$$

Switch to case II when $u(0) = 0 \rightarrow b_{crit} = \sqrt{\dfrac{6Du_b}{Q}}$

Case II : $b > b_{crit}$, $u(b) = u_b$, $u(a) = 0$

$$\rightarrow \quad u = \begin{cases} 0 & r < a \quad \leftarrow \text{"necrotic core"} \\ \dfrac{Qr^2}{6D} + \dfrac{Q(b^3-a^3)}{} \cdots & a < r < b \end{cases}$$

($\nwarrow$ in notes!)

and a is det'd by $u'(a) = 0$

$\rightarrow$ polynomial $\quad \dfrac{Q}{D}\left(2a^3 - 3a^2 b + b^3\right) - 6b\,u_b = 0$

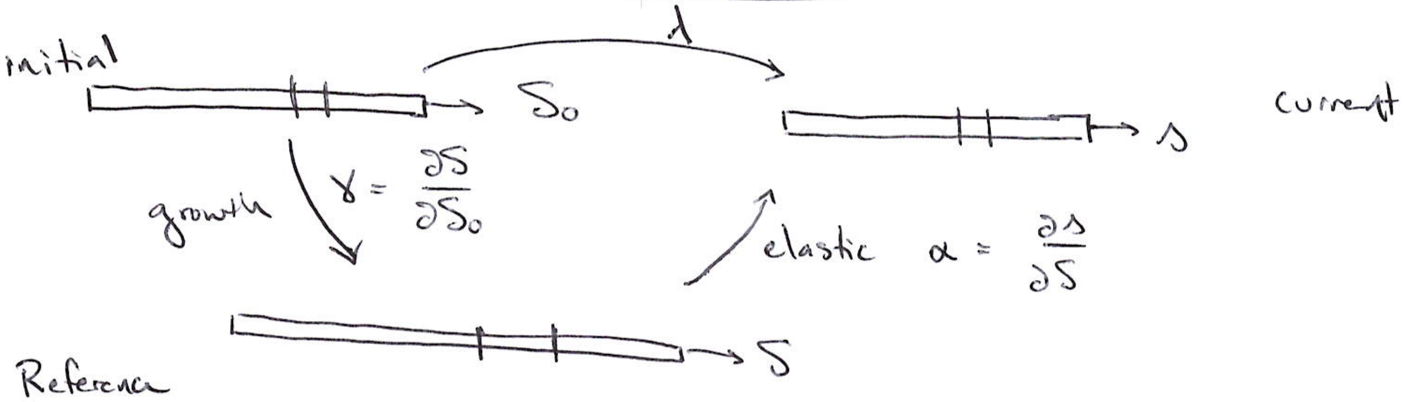
Back to growth ...

$$r^2\,dr = \eta R_0^2\,dR_0 \implies \frac{b^3}{3} = \int_0^{B_0} \eta R_0^2\,dR_0$$

$\frac{\partial}{\partial t}$

$$\implies b^2 \frac{db}{dt} = \int_0^{B_0} \eta_t R_0^2\,dR_0 = k \int_0^{B_0} u\,\underbrace{\eta R_0^2\,dR_0}_{r^2 dr} = k \int_0^b u(r,b)\,r^2\,dr$$

Growth w/ Elastic Response



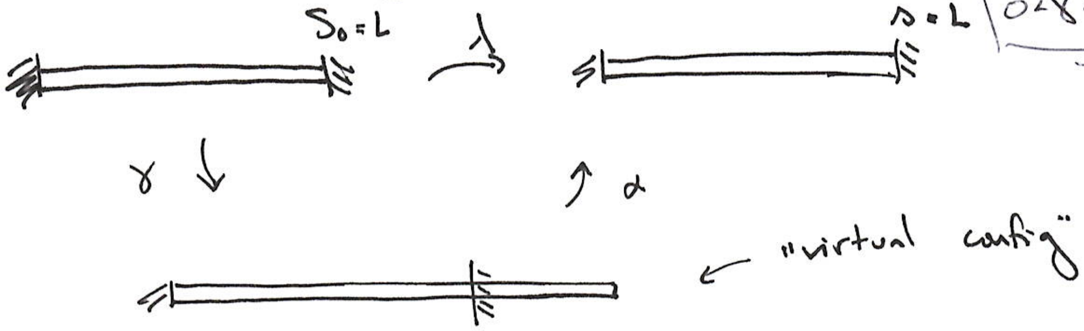
Stretch $\lambda = \frac{\partial s}{\partial S_0} = \frac{\partial s}{\partial S}\frac{\partial S}{\partial S_0} = \alpha\gamma$ "1D Morphoelasticity"

★ Must have $\gamma, \alpha > 0$ ★

$\gamma > 1 \rightarrow$ Growth

$0 < \gamma < 1 \rightarrow$ 'Resorption'

Simple Ex - A rod b/t walls



$\lambda \equiv 1 \Rightarrow \alpha = \frac{1}{\gamma}$. Supp. $\gamma = 1 + t$

Hookean : Stress $\sigma = E(\alpha - 1) = \frac{-Et}{1+t}$

$\sigma \rightarrow -E$ as $t \rightarrow \infty$. Infinite compression but only finite stress!!

Better : neo Hookean $\sigma = \frac{E}{3}\left(\alpha^2 - \frac{1}{\alpha}\right) = \frac{E}{3}\left(\frac{1}{(1+t)^2} - (1+t)\right)$

Now $\sigma \sim -\frac{E}{3}t$ as $t \rightarrow \infty$

Stress dependent growth

$$\frac{\partial \gamma}{\partial t} = \gamma(\sigma - \sigma^*) \quad , \quad \sigma^* \text{ is } \underline{\text{homeostatic}} \text{ ('target') stress}$$

Ex Hookean material $\sigma = E(\alpha - 1)$

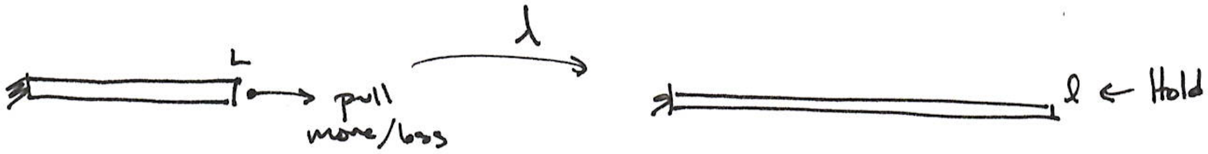
($\Rightarrow$ homeostatic strain $\alpha^* = \frac{\sigma^*}{E} + 1$)

$$\lambda_0 = \frac{l_0}{L} = \alpha_0 = \alpha^*$$

An expt: 1.

Material is 'pulled' to homeostatic stress

2.



At $t=0$, length changed to $l \neq l_0$ and held

Q. How grow to recover homeostatic stress?

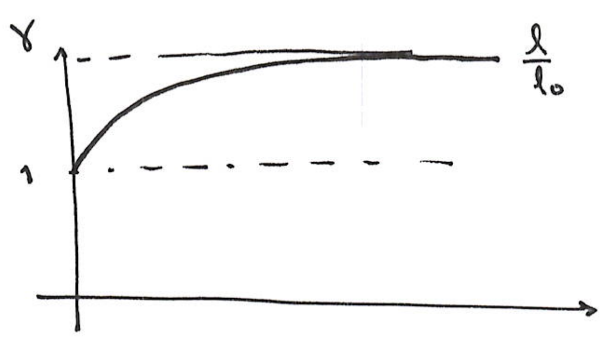
$$\lambda = \frac{l}{L} = \frac{l}{l_0}\lambda_0 = \frac{l}{l_0}\alpha^* \quad \text{is fixed}$$

$$\Rightarrow \alpha = \frac{\lambda}{\gamma} = \frac{l}{l_0 \gamma}\left(\frac{\sigma^*}{E} + 1\right)$$

$\sigma = E(\alpha - 1)$, $\sigma^* =$

& $\sigma = E(\alpha - 1) \Rightarrow \dot{\gamma} = \gamma(\sigma - \sigma^*) = \frac{l}{l_0}(\sigma^* + E) - (\sigma^* + E)\gamma$

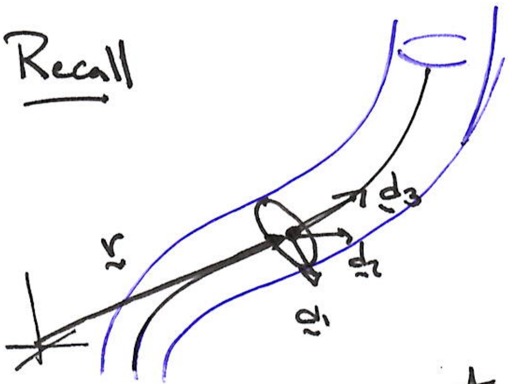
solve w/ $\gamma(0) = 1 \rightarrow \gamma(t) = \frac{l}{l_0} + \left(1 - \frac{l}{l_0}\right)e^{-(\sigma^* + E)t}$



A Growing Elastic Rod

Idea: Extend the framework of elastic rods to include growth

Recall



We had (unshearable, extensible): in equil.

$$\star \begin{cases} \dfrac{\partial \underset{\sim}{r}}{\partial S} = \alpha\, \underset{\sim}{d}_3 \\[2ex] \dfrac{\partial \underset{\sim}{d}_i}{\partial S} = \underset{\sim}{u} \wedge \underset{\sim}{d}_i \\[2ex] \dfrac{\partial \underset{\sim}{n}}{\partial S} + \underset{\sim}{f} = \underset{\sim}{0} \qquad \text{(Force bal.)} \\[2ex] \dfrac{\partial \underset{\sim}{m}}{\partial S} + \dfrac{\partial \underset{\sim}{r}}{\partial S} \wedge \underset{\sim}{n} + \underset{\sim}{l} = \underset{\sim}{0} \qquad \text{(Mom. bal.)} \end{cases}$$

Plus constit:

$$\star\star \begin{cases} \underset{\sim}{m} = EI_1 (u_1 - \hat{u}_1)\, \underset{\sim}{d}_1 + EI_2 (u_2 - \hat{u}_2)\, \underset{\sim}{d}_2 + \mu J (u_3 - \hat{u}_3)\, \underset{\sim}{d}_3 \\[2ex] n_3 = \underset{\sim}{n} \cdot \underset{\sim}{d}_3 = EA(\alpha - 1) \end{cases}$$

Above, $\alpha = \dfrac{\partial s}{\partial S}$. w/ S ref. arclength and s current arclength.

To incorporate axial growth, we ~~decompose~~ introduce fixed init. config w/ arclength S_0, & grown ref. config w/ arclength S st $\gamma = \dfrac{\partial S}{\partial S_0}$ and decompose

$\lambda = \dfrac{\partial s}{\partial S_0} = \alpha \gamma$ as before

The eq's (☆), (☆☆) don't change, but the domain $S \in [0, L]$ will change for $\gamma \neq 1$.

- Can also be cast in S_0 or s variable:

eg in S_0 we'd write $\frac{\partial \underset{\sim}{r}}{\partial S_0} = \alpha \gamma \underset{\sim}{d}_3$,

$$\frac{\partial \underset{\sim}{n}}{\partial S_0} + \gamma \underset{\sim}{f} = \underset{\sim}{0} \quad , \text{ etc}$$

[assumed in (☆) that $\underset{\sim}{f}$ is $\frac{\text{Force}}{\text{Ref length } \Delta S}$

$$\Rightarrow \gamma \underset{\sim}{f} = \frac{\text{Force}}{\Delta S_0} \quad]$$

- Cross-sectional growth would only change the parameters I_1, I_2, J, A – eg could make these fns of time

eg circular cross-section w/ radius $R = \varepsilon(1+t)$

$$I_1 = I_2 = \frac{\pi R^4}{4} \sim \varepsilon t^4, \quad A = \pi R^2 \sim \varepsilon t^2$$

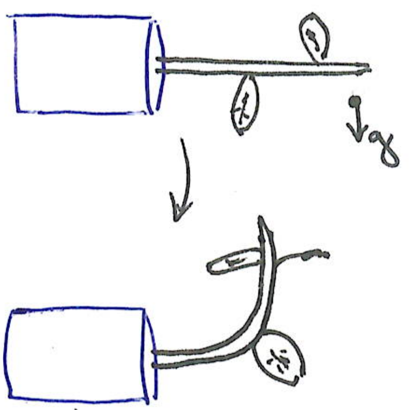
$\rightarrow$ rod stiffens over time

<u>Remodelling</u> - A change in properties w/out a change in mass

eg a change in $\hat{\underset{\sim}{u}}$ intrinsic curvature

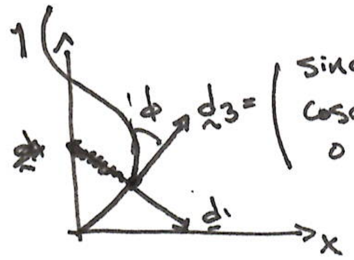
Application : Gravitropism

Expt : put a potted plant on its side



- The plant "wants to be" aligned w/ gravity
→ changes intrinsic curvature

A simple model - planar rod



$$\underset{\sim}{d_3} = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

$$\underset{\sim}{d_1} = \begin{pmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{pmatrix} \qquad \underset{\sim}{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\underset{\sim}{d_2} = -\underset{\sim}{e_z}$$

$$\underset{\sim}{u} = u_2 \underset{\sim}{d_2} \quad \text{w/} \quad u_2 = \kappa = \frac{d\phi}{ds}$$

<u>Geom</u>

$x' = \sin\phi$ ~~$\cos\phi$~~

$y' = \cos\phi$

$\underset{\sim}{d_3}' = u_2 \underset{\sim}{d_1}$

$\underset{\sim}{d_1}' = -u_2 \underset{\sim}{d_3}$

Remodelling : supp. $\frac{\partial \hat{u}_2}{\partial t} = -\beta \sin\phi$

"negative gravit."

Kinematics only : (No mechanics)

$\underset{\sim}{u} \equiv \hat{\underset{\sim}{u}}$, so $\begin{cases} \phi' = u_2 \\ \dot{u}_2 = -\beta \sin\phi \end{cases}$ determines shape evol'n

If $\underline{\text{nearly vertical}}$: take $|\phi| \ll 1$

$x' \approx \phi \quad \longrightarrow \quad x_{sst} + \beta x_s = 0$

$y' \approx 1 \quad \Rightarrow \quad x_{st} + \beta x = c(t)$

$\phi' = u_2$

$\dot{u}_2 \approx -\beta\phi$

$\underline{\text{BC}}$ At $s=0$, clamp at angle $\phi_0 \ll 1$

$\longrightarrow \quad x(0,t) = 0 \; , \; x'(0,t) = \phi_0$

$\Downarrow$

$x_{st}(0,t) = 0$

$\Downarrow$

$c = 0$

$$\Rightarrow \begin{cases} x_{st} + \beta x = 0 & (1) \\ x(0,t) = 0 & (2) \\ x(s,0) = \phi_0 s & (3) \end{cases}$$

init. cond. if $\phi(s,0) \equiv \phi_0$

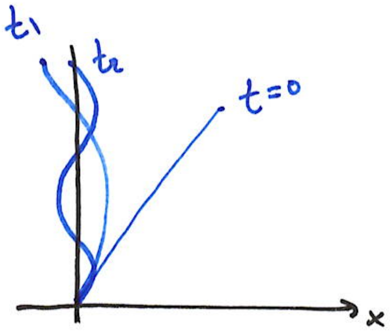
Can solve as similarity soln (B5.2 !)

seek $x = s^\alpha f(\underbrace{st}_{\eta})$, then (3) $\Rightarrow \alpha = 1$, $f(0) = \phi_0$

& (1) $\rightarrow \eta f''(\eta) + 2f'(\eta) + \beta f = 0$

which has soln $f = \phi_0 \dfrac{J_1(2\sqrt{\beta\eta})}{\sqrt{\beta\eta}}$ — Bessel fn 1st kind

$$\Rightarrow \quad x(s,t) = \phi_0 s \frac{J_1(2\sqrt{\beta st})}{\sqrt{\beta st}}$$



Mechanical Pattern Formation

"Trivial state" $\xrightarrow[\text{Growth + Constraint}]{}$ Pattern (complex state)

Compare: A) in a biochemical pattern (Turing pattern), concentrations of chemicals go from a homogeneous state to a patterned state due to reaction, diffusion

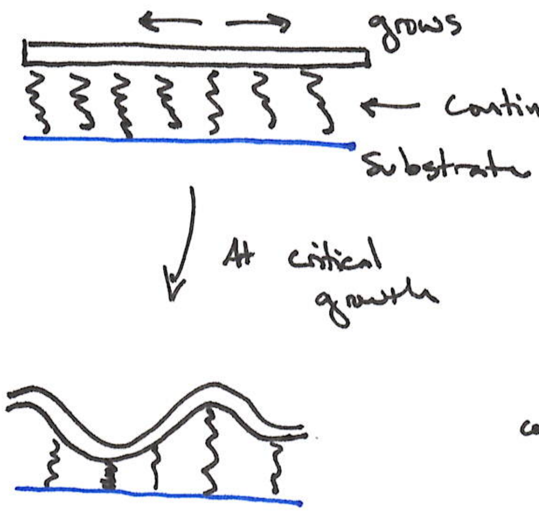
B) A biomechanical pattern is structural, ie a material deforms from a base state (flat, eg) to a patterned state

- Note - both types may be present and linked!

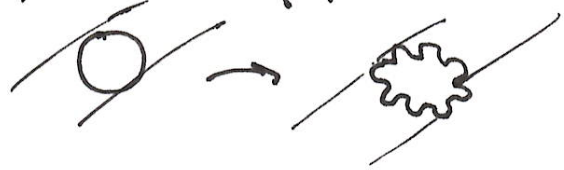
Ex Wrinkling instability - Rod on Foundation

Ingredients : a growing elastic beam (or sheet) attached to a substrate ('foundation')

an external constraint



Basic idea ~~[illegible]~~ underlying morphogenesis of many patterns - wrinkles in skin, cortical folds in brain, ~~spi~~ ridges/spines on seashells, ~~[illegible]~~ shape of airways, intestine



Growing beam : $$\underset{\sim}{r} = x(S_0)\,\underset{\sim}{e}_x + y(S_0)\,\underset{\sim}{e}_y$$

$$\frac{d\underset{\sim}{r}}{dS_0} = \alpha\gamma\,\underset{\sim}{d}_3 \quad \text{w/} \quad \underset{\sim}{d}_3 = \cos\theta(S_0)\,\underset{\sim}{e}_x + \sin\theta(S_0)\,\underset{\sim}{e}_y$$

Geom.

$$\frac{d\underset{\sim}{d}_3}{dS_0} = \theta'(S_0)\,\underset{\sim}{d}_1 \quad \text{w/} \quad \underset{\sim}{d}_1 = -\sin\theta\,\underset{\sim}{e}_x + \cos\theta\,\underset{\sim}{e}_y$$

Let $\underset{\sim}{n} = F\,\underset{\sim}{e}_x + G\,\underset{\sim}{e}_y$, force due to substrate :

$\underset{\sim}{f} = f\,\underset{\sim}{e}_x + g\,\underset{\sim}{e}_y$, and moment $\underset{\sim}{m} = m\,\underset{\sim}{e}_z$

Force and mom balance:

$$\frac{dF}{dS_0} + f = 0 \ , \quad \frac{dG}{dS_0} + g = 0 \ , \quad \frac{dm}{dS_0} + \alpha\gamma\left(G\cos\theta - F\sin\theta\right) = 0$$

Constit:

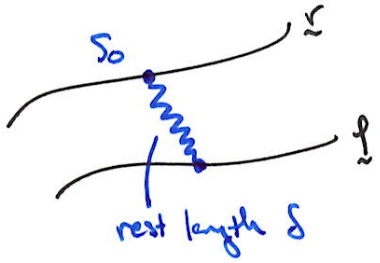
$$m = EI\frac{d\theta}{dS} = \frac{EI}{\gamma}\frac{d\theta}{dS_0}$$

& $\quad \underset{\sim}{n} \cdot \underset{\sim}{d}_3 = F\cos\theta + G\sin\theta = EA(\alpha - 1)$

The foundation: define a curve $\underset{\sim}{\rho} = \rho_x(S_0)\underset{\sim}{e}_x + \rho_y(S_0)\underset{\sim}{e}_y$

and make a 1-1 map 'glueing' $\underset{\sim}{\rho}$ to $\underset{\sim}{r}$ w/

~~linear~~ springs



let $\Delta(S_0) = \| \underset{\sim}{\rho}(S_0) - \underset{\sim}{r}(S_0) \|$

Then

$$\underset{\sim}{f} = h(\Delta - \delta)\left(\frac{\underset{\sim}{r} - \underset{\sim}{\rho}}{\Delta}\right)$$

Simplest: $\underset{\sim}{\rho} = S_0\underset{\sim}{e}_x$ (& before growth $\underset{\sim}{r} = S_0\underset{\sim}{e}_x$) $\Rightarrow \delta = 0$

If $h(0) = 0$, $h'(0) = -k$, then $\underset{\sim}{f} \approx -k\left((x - S_0)\underset{\sim}{e}_x + y\underset{\sim}{e}_y\right)$

$$\rightarrow \quad \frac{dF}{dS_0} = k(x - S_0) \ , \quad \frac{dG}{dS_0} = ky$$

Observe: possible to have growth ($\gamma > 1$) w/out deformation:

$$\lambda = \alpha\gamma = 1 \rightarrow \alpha = \frac{1}{\gamma}$$

$$x = S_0, \quad y = 0, \quad \theta = 0, \quad m = 0, \quad G = 0$$

$$F = EA(\alpha - 1) = EA\left(\frac{1-\gamma}{\gamma}\right)$$

"Compressed but still flat"

- Pattern forms when compression gets too high

"a tradeoff of bending & foundation energy to relieve compressive energy"

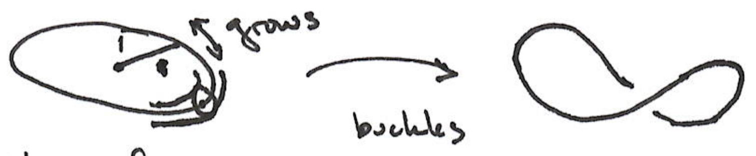
When? Depends on stiffness of substrate (via k)

Buckling analysis: $x = S_0 + \varepsilon x_1, \quad \theta = \varepsilon\theta_1, \quad \ldots$

(Problem sheets)

- Find crit. γ for which linearised system has a sol'n

Ex. A growing ring (inextensible)



stress-free

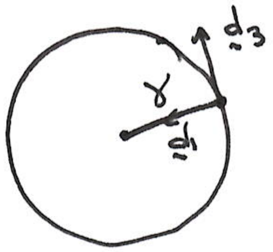
$u_2 = \hat{u}_2 = 1$

- simple model for shape of potato chip, various leaves, algae

- Here the constraint is internal, due to the closed geometry

- Pre-buckled state:

$s \in (0, 2\pi\gamma)$



$$\underset{\sim}{d}_1 = -\begin{pmatrix} \cos\frac{s}{\gamma} \\ \sin\frac{s}{\gamma} \\ 0 \end{pmatrix}, \quad \underset{\sim}{d}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{d}_3 = \begin{pmatrix} -\sin s/\gamma \\ \cos s/\gamma \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\gamma}, \quad u_1 = u_3 = 0$$

$$\rightarrow \underset{\sim}{m} = EI_2\left(\frac{1}{\gamma} - 1\right)\underset{\sim}{d}_2, \quad \underset{\sim}{n} = \underset{\sim}{0} \quad \leftarrow \left(\text{in this problem } \underset{\sim}{f} = \underset{\sim}{0}\right)$$

Buckling: again seek sol'n nearly flat, ie $u_1 = \varepsilon u_1^{(1)}$,
$u_2 = \frac{1}{\gamma} + \varepsilon u_2^{(1)}$, etc..

Harder ~~Further~~ because buckled shape is 3D!
($\Rightarrow$ eg $\underset{\sim}{d}_i$ perturb as well!)

A trick: expand $\underset{\sim}{d}_i = \underset{\sim}{d}_i^{(0)} + \varepsilon\, \underset{\sim}{c} \wedge \underset{\sim}{d}_i^{(0)} + O(\varepsilon^2)$, and for each variable $\underset{\sim}{v} = \sum v_i \underset{\sim}{d}_i$, expand as $\underset{\sim}{v} = \sum \left(v_i^{(0)} + \varepsilon v_i^{(1)}\right)\underset{\sim}{d}_i^{(0)}$

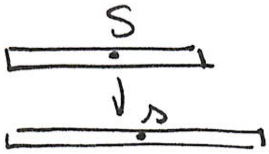
(Problems...)

3D Growth

1. Review / Summary of nonlinear elasticity
2. Build-in growth → "Morphoelasticity"

Nonlin elasticity in 10 EASY Steps:

(0.) Recall 1D :



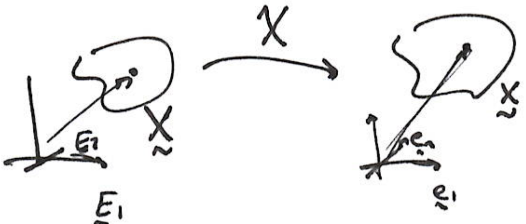
$\frac{ds}{dS} = \alpha$ strain

$n' + f = \rho \ddot{s}$ FB

internal "stress" — external forces

$n = \hat{n}(\alpha)$ CL

(1) Kinematics - continuous deformation of body $\mathcal{B}_0$ to $\mathcal{B}_t$



$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t)$$

(2) Tensors $\underset{\sim}{u} \otimes \underset{\sim}{v}$ tensor product defined by

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})\underset{\sim}{a} = (\underset{\sim}{v} \cdot \underset{\sim}{a})\underset{\sim}{u}$$

A tensor $T = T_{ij}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$, $T_{ij} = T\underset{\sim}{e_j} \cdot \underset{\sim}{e_i}$

Let $\phi(\underset{\sim}{x})$ scalar, $\underset{\sim}{u}$ vec, $= u_i\, \underset{\sim}{e_i}$, $T = T_{ij}\, e_i \otimes e_j$ Tensor

$\rightarrow \operatorname{grad}\phi = \frac{\partial \phi}{\partial x_i} \underset{\sim}{e_i}$, $\operatorname{grad}\underset{\sim}{u} = \frac{\partial u_i}{\partial x_j} \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$

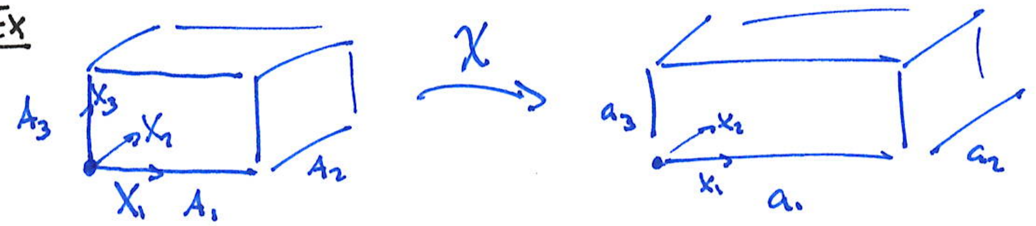
$\operatorname{div} T = \frac{\partial T_{ij}}{\partial x_i} \underset{\sim}{e_j}$, if $\underset{\text{scalar}}{A} = A(\underset{\text{tensor}}{F})$, $\frac{\partial A}{\partial F} = \frac{\partial A}{\partial F_{ji}} \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$

(3) Deformation gradient

$$F = \text{Grad}\, \underset{\sim}{x} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = \frac{\partial x_i}{\partial X_j} \underset{\sim}{e}_i \otimes \underset{\sim}{E}_j$$

Then $J = \det F > 0$ gives volume change ~~[illegible]~~ $dv = J\,dV$

<u>Ex</u>



$$\underset{\sim}{x} = \sum \frac{a_i}{A_i} X_i \underset{\sim}{e}_i \rightarrow F = \text{diag}\left(\frac{a_1}{A_1}, \frac{a_2}{A_2}, \frac{a_3}{A_3}\right)$$

$$= \sum_{i=1}^{3} \frac{a_i}{A_i} \underset{\sim}{e}_i \otimes \underset{\sim}{E}_i$$

(4) Force balance (lin. momentum)

$$\frac{d}{dt} \int_{\Omega} \rho \underset{\sim}{v}\, dv = \int_{\Omega} \rho \underset{\sim}{b}\, dv + \int_{\partial\Omega} \underset{\sim}{t}\, dA$$

density vel. body force contact force

Cauchy : $\exists$ tensor T st $\underset{\sim}{t} = T\underset{\sim}{n}$ for unit normal $\underset{\sim}{n}$

Ω arb. + div thm $\rightarrow$ $\text{div}\, T + \rho \underset{\sim}{b} = \rho \dot{\underset{\sim}{v}}$

Angul. mom'm : $T^T = T$

(5) Hyperelastic material : $\exists$ strain-energy fn $W = W(F)$

$\rightarrow$ elastic energy is $\int_{\mathcal{B}} W\, dV$

(6) Energy balance $\rightarrow$ $T = J^{-1} F \frac{\partial W}{\partial F}$ compressible

$T = F\frac{\partial W}{\partial F} - p\mathbb{1}$ incomp. ($J \equiv 1$, p Lagrange mult. - hydrostatic pressure)

" Discuss: basically have all ingredients of 1D, but can't use it: what is $W(F)$?

∘ in 1D it's $n = h(\alpha)$ ← stretch! "

⑦ Stretches

$d\underset{\sim}{X} = \underset{\sim}{M}\, dS$ → $d\underset{\sim}{x} = \underset{\sim}{m}\, ds$

($\underset{\sim}{M}$: unit vec; $\underset{\sim}{m}$: unit)

$$d\underset{\sim}{x} = F d\underset{\sim}{X} \Rightarrow \underset{\sim}{m}\, ds = F\underset{\sim}{M}\, dS \underset{\text{norm}}{\Rightarrow} |ds|^2 = (F\underset{\sim}{M})\cdot(F\underset{\sim}{M})\, |dS|^2$$

$$\Rightarrow \text{stretch} \quad \lambda(\underset{\sim}{M}) = \frac{ds}{dS} = \sqrt{(F^T F \underset{\sim}{M})\cdot \underset{\sim}{M}}$$

$F^T F$ ← characterising strain

Material is unstrained (no stretch in any direction) if $F^T F \equiv \mathbb{1}$

⑧ Polar decomp.

$\det F > 0 \Rightarrow \exists$ unique U, V symm, pos. def and orthog R st

↳ $R^T R = \mathbb{1}$, $\det R = 1$

- $F = RU = VR$
- Then $\quad F^T F = U^2 =: C \qquad$ Right Cauchy-Green tensor

$\qquad\quad FF^T = V^2 =: B \qquad$ Left " "

- Can write $\quad V = \sum_{i=1}^{3} \lambda_i\, \underset{\sim}{v}_i \otimes \underset{\sim}{v}_i$

$\{\lambda_i, \underset{\sim}{v}_i\}$ are eig vals, vecs of V. λ_i positive and real since V pos. symm.

- principal stretches (λ_i) and directions $\underset{\sim}{v}_i$ of def.

(U has same λ_i)

Note $J = \det F = \det V = \lambda_1 \lambda_2 \lambda_3$

(9) Isotropic material (same response in any direction)

$$W(F) = W(V)$$

$\uparrow$ rotations don't matter!

- Frame invariance (objectivity) $\Rightarrow$ W only depends on principal invariants of V

OR, more convenient, express W through invariants of $B = V^2 = FF^T$ (coeff's of charact. poly):

$$I_1 = \operatorname{tr} B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \operatorname{tr}(B^2)\right) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2$$

$$I_3 = \det B = \lambda_1^2\lambda_2^2\lambda_3^2$$

$\hookrightarrow$ so really $W = W(\lambda_1, \lambda_2, \lambda_3)$ (or sometimes $W(I_1, I_2, I_3)$)

- incompressible : $\lambda_3 = \dfrac{1}{\lambda_1\lambda_2}$, $I_3 \equiv 1$

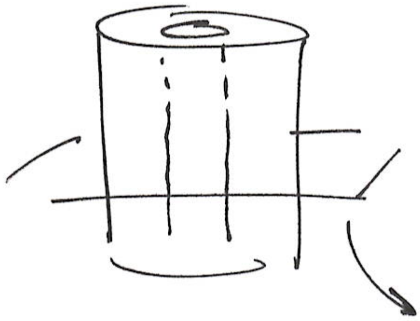
(10) $T = F\dfrac{\partial W}{\partial F} - p\mathbb{1}$ (incomp.)

"How to use all the lin. alg tricks?"

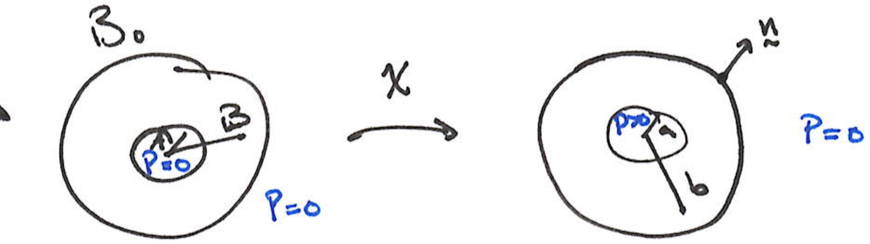
$$\Rightarrow T = \sum t_i \, \underset{\sim}{v}_i \otimes \underset{\sim}{v}_i \quad \text{w/} \quad t_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p$$

[in $\{\underset{\sim}{v}_i\}$ basis, $F = V = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\rightarrow \dfrac{\partial W}{\partial F} = \operatorname{diag}\left(\dfrac{\partial W}{\partial \lambda_1}, \dfrac{\partial W}{\partial \lambda_2}, \dfrac{\partial W}{\partial \lambda_3}\right)$]

Ex Inflation of a cylinder



- Assume: incompressible, no axial def, remains symmetric



$$\underset{\sim}{X} = R\,\underset{\sim}{E}_r \ , \quad \underset{\sim}{x} = r(R)\,\underset{\sim}{e}_r \ , \quad \theta = \Theta$$

Then
$$F = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = r'(R)\,\underset{\sim}{e}_r \otimes \underset{\sim}{E}_r + \frac{r}{R}\,\underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta = \mathrm{diag}\Big(\underbrace{r'}_{\lambda_r},\ \underbrace{\frac{r}{R}}_{\lambda_\theta},\ 1\Big)$$

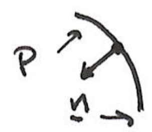
- incomp $\Rightarrow \lambda_\theta = \dfrac{1}{\lambda_r} \Rightarrow \quad r\,dr = R\,dR \Rightarrow r^2 - a^2 = R^2 - A^2$

* deformation fully det'd once a known

- Force bal: $\operatorname{div} T = 0 \Rightarrow \dfrac{dt_r}{dr} + \dfrac{t_r - t_\theta}{r} = 0$ $\quad \left(T = t_r\,\underset{\sim}{e}_r \otimes \underset{\sim}{e}_r + t_\theta\,\underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta \right)$

- Bd'y cond: $T\underset{\sim}{n} \cdot \underset{\sim}{n} = t_r = 0$ at $r = b$

  $t_r = -P$ at $r = a$



- Constit: $t_r = \lambda_r \dfrac{\partial W}{\partial \lambda_r} - p \ , \quad t_\theta = \lambda_\theta \dfrac{\partial W}{\partial \lambda_\theta} - p$ ← unknown - different p to BC!

$$\Rightarrow \frac{dt_r}{dr} = \frac{\lambda_\theta \frac{\partial W}{\partial \lambda_\theta} - \lambda_r \frac{\partial W}{\partial \lambda_r}}{r} \qquad // \int_a^b dr$$

$$\boxed{\Rightarrow P = \int_a^b \frac{\lambda_\theta \frac{\partial W}{\partial \lambda_\theta} - \lambda_r \frac{\partial W}{\partial \lambda_r}}{r}\,dr}$$

← given $P, A, B, W(\lambda_r, \lambda_\theta)$, this is an eq'n for a

$$T = t_r \, \underset{\sim}{e}_r \otimes \underset{\sim}{e}_r + t_\theta \, \underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta$$

$$\underset{\sim}{e}_r = \cos\theta \, \underset{\sim}{e}_1 + \sin\theta \, \underset{\sim}{e}_2 \qquad \underset{\sim}{x} = r\cos\theta \, \underset{\sim}{e}_1 + r\sin\theta \, \underset{\sim}{e}_2$$

$$= \frac{\partial \underset{\sim}{x}}{\partial r}$$

$$\underset{\sim}{e}_\theta = \frac{1}{r} \frac{\partial \underset{\sim}{x}}{\partial \theta} \qquad \underset{\sim}{e}_1 = \frac{1}{h_1} \frac{\partial \underset{\sim}{x}}{\partial r} + \frac{1}{h_2} \frac{\partial \underset{\sim}{x}}{\partial \theta}$$

$$\operatorname{div} T = \frac{\partial T_{ij}}{\partial x_i} \underset{\sim}{e}_j = \frac{\partial T_{11}}{\partial x_1} \underset{\sim}{e}_1 + \frac{\partial T}{}$$

$$(\underset{\sim}{u} \otimes \underset{\sim}{v}) \underset{\sim}{a} = (\underset{\sim}{v} \cdot \underset{\sim}{a}) \underset{\sim}{u}$$

$$\operatorname{div} T = \frac{\partial T}{\partial r} \cdot \underset{\sim}{e}_r + \frac{1}{r} \frac{\partial T}{\partial \theta} \cdot \underset{\sim}{e}_\theta$$

$$= \frac{\partial}{\partial r}\left(t_r \, \underset{\sim}{e}_r \otimes \underset{\sim}{e}_r\right) \cdot \underset{\sim}{e}_r + \frac{\partial}{\partial r}\left(t_\theta \, \underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta\right) \cdot \underset{\sim}{e}_r$$

$$+ \frac{1}{r} \frac{\partial}{\partial \theta}\left(t_r \, \underset{\sim}{e}_r \otimes \underset{\sim}{e}_r + t_\theta \, \underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta\right) \cdot \underset{\sim}{e}_\theta$$

$$\frac{\partial \underset{\sim}{e}_r}{\partial \theta} = \underset{\sim}{e}_\theta$$

$$= \left(\frac{\partial t_r}{\partial r} + \frac{1}{r}\left(t_r - t_\theta\right)\right) \underset{\sim}{e}_r$$

$$\frac{\partial \underset{\sim}{e}_\theta}{\partial \theta} = -\underset{\sim}{e}_r$$

eg neo-Hookean (standard, "easiest" nonlin)

$$W = \frac{\mu}{2}(I_1 - 3) = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 1\right)$$, $\mu = 3E$ is called shear mod.

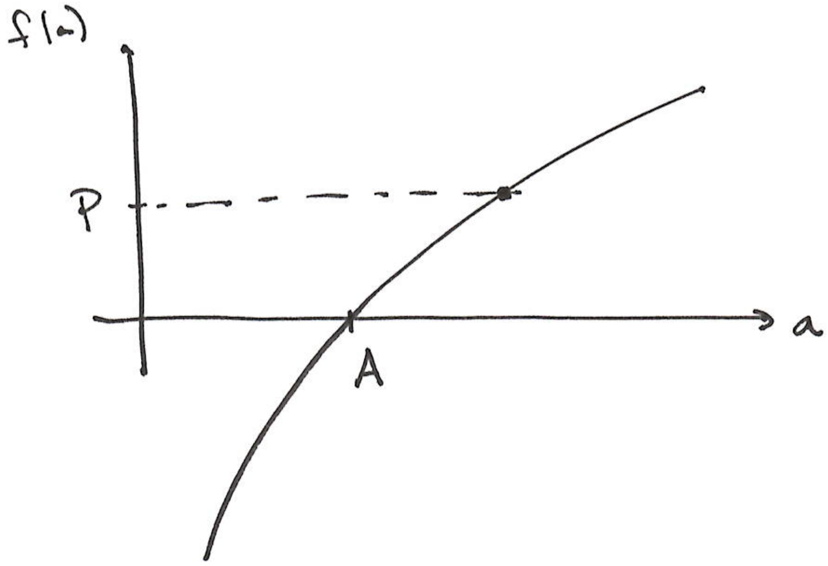
$$\Rightarrow \lambda_i \frac{\partial W}{\partial \lambda_j} = \mu \lambda_i^2$$

$$\rightarrow P = \mu \int_a^b \frac{\lambda^2 - \frac{1}{\lambda^2}}{r}\, dr \quad \text{w/} \quad \lambda := \frac{r}{R}, \ \lambda_r = \frac{1}{\lambda}$$

$$= \mu \int_A^B \frac{\lambda^2 - \lambda^{-2}}{r(R)^2} R\,dR \quad \longleftarrow \quad \lambda = \lambda(R) = \frac{\sqrt{a^2 + R^2 - A^2}}{R}$$

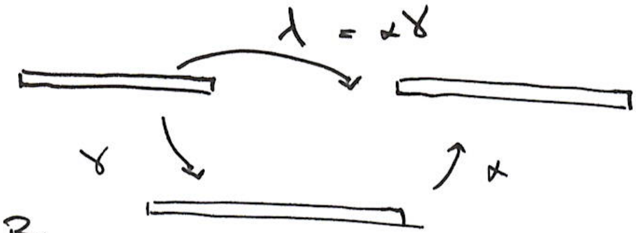
$r\,dr = R\,dR$

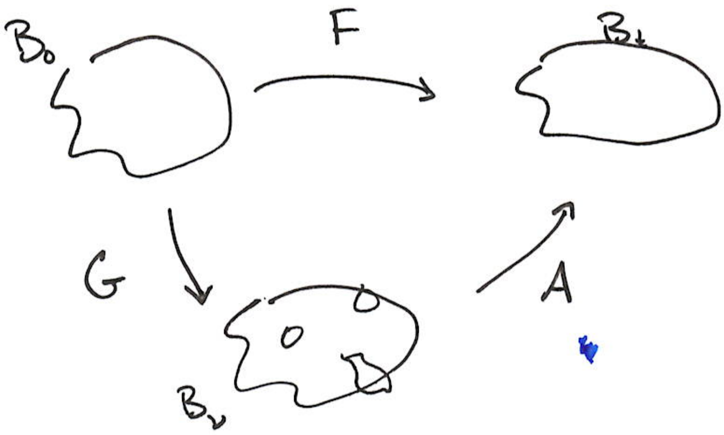
$$=: f(a)$$



Morphoelasticity - a framework for growing elastic bodies

· same idea as 1D!





Multip. decomp.

$$F = A G$$

- G · growth tensor · describes local increase (or decrease) of mass → "virtual config B_v"

  [Discuss · why a tensor? growth can occur diff'tly in diff't directions → so B_v may be incompatible state (w/ holes, eg), but still stress free]

- A · elastic tensor · "restores compatibility"

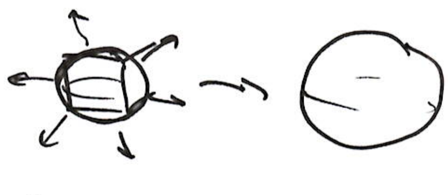
  → stress only depends on elastic def^{n}:

$$T = A \frac{\partial W}{\partial A} - p \mathbb{1}$$

· all other eq'ns same as before!

Types of growth

Isotropic - same in all directions



vs $\quad G = g\mathbb{1} \quad$ ($g > 1$ for growth)

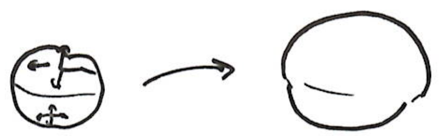
Anisotropic - NOT " "

- eg transversely isotropic - one growth direction $\underset{\sim}{g}$: $G = \gamma\, \underset{\sim}{g} \otimes \underset{\sim}{g}$ "fibre growth"

Homogeneous - same form of growth everywhere

vs

Heterogeneous - growth varies through material

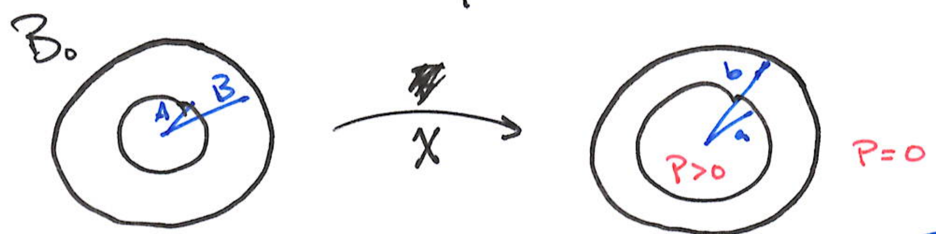


Note - Anisotropy and/or heterogen. creates incompatibility $\Rightarrow$ the current config $\mathcal{B}_t$ may be stressed even if unloaded

- this is called residual stress - very common and important in biological tissues (trees, arteries, skin, ..)

Ex. A Growing Cylinder

- As before, assume incompressible, no axial def, symmetric.



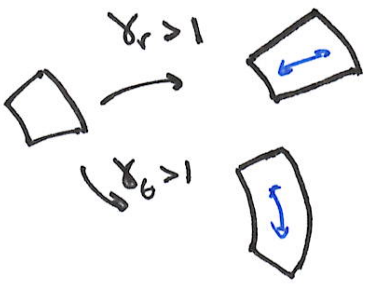
radial growth

circumf. gr.

$$F = \mathrm{Grad}\,\chi = AG \quad \text{w/} \quad G = \mathrm{diag}(\gamma_r, \gamma_\theta, 1)$$

$$A = \mathrm{diag}(\alpha_r, \alpha_\theta, 1)$$

$$F = \mathrm{diag}\left(r'(R), \frac{r}{R}, 1\right)$$



- $F = AG \Rightarrow \quad r' = \alpha_r \gamma_r, \quad \frac{r}{R} = \alpha_\theta \gamma_\theta$

- Incomp. $\Rightarrow \quad \det A = 1 \quad \Rightarrow \quad \alpha_r = \frac{1}{\alpha_\theta} =: \frac{1}{\alpha}$

$$\Rightarrow \frac{r}{\gamma_\theta R} = \frac{\gamma_r}{r'} \Rightarrow r\,r'(R) = \gamma_r \gamma_\theta R$$

$$\Rightarrow r^2 - a^2 = \int_A^R \gamma_r(\rho)\gamma_\theta(\rho)\,\rho\, d\rho$$

[given γ_r, γ_θ, need to find a]

- Force balance $\quad \mathrm{div}\,T = 0 \rightarrow \frac{dt_r}{dr} + \frac{t_r - t_\theta}{r} = 0$

- Bdy cond $\quad t_r = 0$ at $r = b$

$\quad t_r = -P$ at $r = a$

• Constit : $t_r = \alpha_r \frac{\partial W}{\partial \alpha_r} - p$, $t_\theta = \alpha_\theta \frac{\partial W}{\partial \alpha_\theta} - p$

$T = A \frac{\partial W}{\partial A} - p\mathbb{1}$

as before $\longrightarrow$ $$P = \int_a^b \frac{\alpha_\theta \frac{\partial W}{\partial \alpha_\theta} - \alpha_r \frac{\partial W}{\partial \alpha_r}}{r} \, dr$$ ← an eq'n to determine a, given $\{A, B, P, \gamma_r(R), \gamma_\theta(R), W\}$

• eg neo-Hookean $W = \frac{\mu}{2}\left(\alpha_r^2 + \alpha_\theta^2 + \alpha_z^2 - 1 \right)$

$$\rightarrow P = \mu \int_a^b \frac{\alpha^2 - \frac{1}{\alpha^2}}{r} \, dr$$

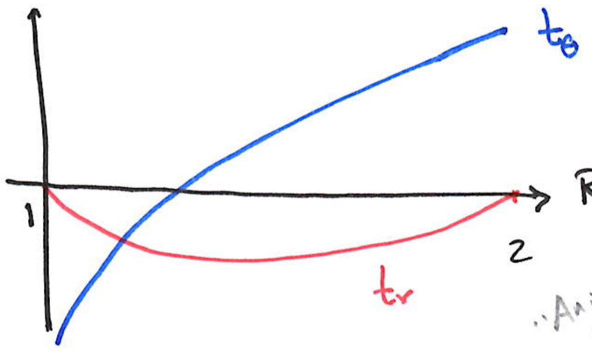
But $\alpha = \frac{r(R)}{\gamma_\theta R}$

and $r\,dr = \gamma_r \gamma_\theta R \, dR$

$$= \mu \int_A^B \frac{\alpha(R)^2 - \alpha(R)^{-2}}{r(R)^2} \gamma_r(R) \gamma_\theta(R) \, R \, dR$$

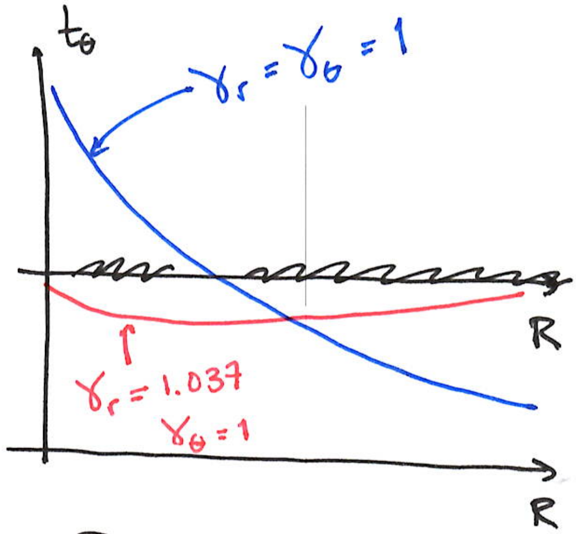
(Discuss - what if $\gamma_r = \gamma_\theta = \text{const}$?)

Ex $A = 1$, $B = 2$, $\mu = 1$



"Anisotropic, homog."

$P = 0$, $\gamma_r = 3$, $\gamma_\theta = 2$



$P = .1$

• t_θ "hoop stress"

stretched

compressed

$t_\theta(2) > 0$: circumferential tension on outside

$t_\theta(1) < 0$: " " compression on "inside"

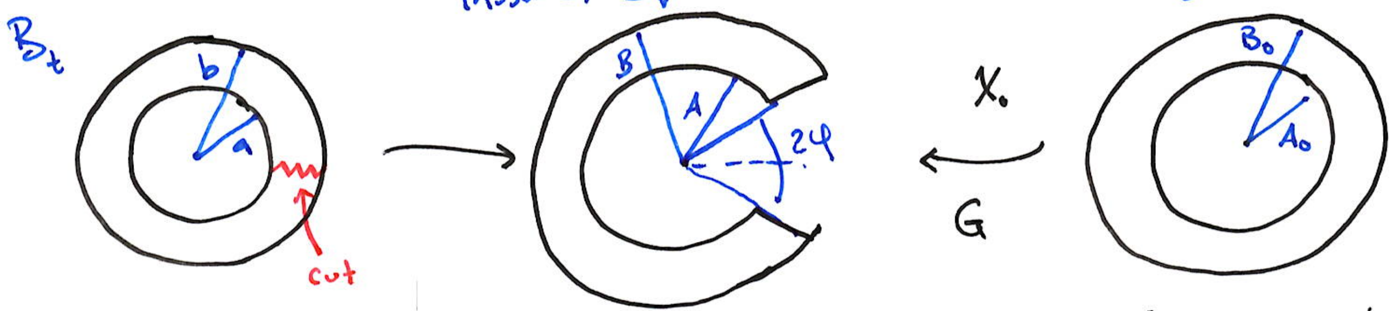
• residual stress can reduce stress gradients!

How to measure residual stress?

- In practice, we usually don't know G, and we only have access to $\mathcal{B}_t$ (current) – but whole framework relies on knowing $\mathcal{B}_0$, $\mathcal{B}_v$!

- Possible resolution – determine G by relieving residual stress

Ex. Opening angle for residually stressed artery



$\{r, \theta, z\}$ $\{R, \Theta, Z\}$ $\{R_0, \Theta_0, Z_0\}$

Supp. for simplicity the map from $\mathcal{B}_0$ to $\mathcal{B}_v$ doesn't change radius or length: $R = R_0$, $Z = Z_0$, $\Theta = \varphi + \frac{2\pi - 2\varphi}{2\pi}\Theta_0$

Then $G = \text{Grad}\, \chi_0 = \text{diag}\left(1, 1 - \frac{\varphi}{\pi}, 1\right)$ * $\vec{\text{covers}}$

Thus $\gamma_r = 1$, $\gamma_\theta = 1 - \frac{\varphi}{\pi}$, and now can compute stress, etc as before