Problem Sheet 2: Suggested Answers and Hints

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TO BE HANDED IN BY FRIDAY OF WEEK 5: questions 8, 9 from Module I and questions 1, 3, 5 from Module II

Module I

Question 8 - Axon Injury

We begin with the following equation:

$$B\frac{d^4W}{dx^4} + P\frac{d^2W}{dx^2} + kW = 0, \tag{1}$$

and non-dimensionalize it by including a characteristic length-scale, L, and non-dimensionalized variable, $x = XL$. In this instance, we find:

$$\frac{d^4w}{dX^4} + \lambda\frac{d^2w}{dX^2} + \beta w = 0, \tag{2}$$

where $\lambda = \frac{PL^2}{B}$ and $\beta = \frac{kL^4}{B}$ are non-dimensionalized parameters and we have new boundary conditions $w(\pm 1) = 0$ and $\frac{dw}{dX}|_{X=\pm 1} = 0$.

Using the ansatz $w(X) = e^{i\omega X}$ in the above equation, we find:

$$\omega^4 - \lambda\omega^2 + \beta = 0. \tag{3}$$

Upon using the discriminant (by letting $\alpha = \omega^2$), real solutions will be obtained provided that $\lambda^2 - 4\beta > 0$. This corresponds to the non-trivial solution required in the question, however, you are invited to think about why that might be. What would the solution be if ω is real? What if ω is a repeated root? Imaginary? Hence why is it necessary that $\lambda^2 - 4\beta > 0$?

We can solve the characteristic equation for ω to find 4 real solutions: ω_+, ω_-, $-\omega_+$ and $-\omega_-$.

As such, the general solution to the non-dimensionalized equation is given by:

$$w(X) = A\cos(\omega_+ X) + B\cos(\omega_- X) + C\sin(\omega_+ X) + D\sin(\omega_- X), \tag{4}$$

for A, B (not to be confused with the bending stiffness), C and D being arbitrary constants to be determined using the boundary conditions. Doing this yields two possible scenarios:

$$w(X) = A_0\left(\frac{\cos(\omega_+ X)}{\cos(\omega_+)} - \frac{\cos(\omega_- X)}{\cos(\omega_-)}\right), \tag{5}$$

whereby $A_0 = A\cos(\omega_+)$ and where we have a relationship for the form $F(\omega^+, \omega_1) = 0$, a dispersion relationship that you should work out.

Alternatively, another set of solutions satisfies

$$w(X) = C_0\left(\frac{\sin(\omega_+ X)}{\sin(\omega_+)} - \frac{\sin(\omega_- X)}{\sin(\omega_-)}\right), \tag{6}$$

where $B_0 = B\sin(\omega_+)$ and with a different dispersion relationship $G(\omega^+, \omega_1) = 0$ to be worked out.

Note that A_0 and C_0 are still undefined even after using the boundary conditions. In a biological context, this doesn't make sense; an axon cannot be stretched to infinity. What further constraints can we use to find A_0 or C_0?

Additionally, you may have noticed that P is not defined in the question. We can, however, solve for P using the relationships $\omega_+ \tan(\omega_+) = \omega_- \tan(\omega_-)$ or $\omega_+ \cot(\omega_+) = \omega_- \cot(\omega_-)$, but there are infinitely many solutions in this instance. How do we fix P?

Question 9 - Derivation of Beam Equation

We begin with the general equations for a rod confined to planar motion:

$$\frac{\partial F}{\partial s} + f = \rho A \frac{\partial^2 x}{\partial t^2} \tag{7}$$

$$\frac{\partial G}{\partial s} + g = \rho A \frac{\partial^2 y}{\partial t^2} \tag{8}$$

$$EI\frac{\partial^2 \theta}{\partial s^2} + G\cos\theta - F\sin\theta = \rho I \frac{\partial^2 \theta}{\partial t^2}, \tag{9}$$

where:

$$\frac{\partial x}{\partial s} = \cos\theta \tag{10}$$

$$\frac{\partial y}{\partial s} = \sin\theta. \tag{11}$$

We use the hint from the problem sheet; namely, to consider $\theta \ll 1$. In this case, we can consider the corresponding asymptotic behavior in the limit of $\theta \to 0$:

$$\frac{\partial x}{\partial s} \sim 1 \Rightarrow x = s \tag{12}$$

$$\frac{\partial y}{\partial s} \sim \theta. \tag{13}$$

So equation (9) becomes:

$$EI\frac{\partial^3 y}{\partial x^3} + G - F\theta = 0, \tag{14}$$

Now differentiate once more with respect to x...

Module II

Question 1 - Invariance of Arclength and Area

As with most questions requiring a proof, there are multiple ways to derive the desired result. Again, you are encouraged to work out a method that makes sense to you.

Under our original parametrization, we define our surface metric to be g_{ij} with corresponding displacement coordinates ξ^1 and ξ^2. After a change of parametrization, these now become $g_{ij}^\dagger$, $(\xi^1)^\dagger$ and $(\xi^2)^\dagger$ respectively. For this transformation, we have a Jacobian

$$J = \begin{bmatrix} \frac{\partial(\xi^1)^\dagger}{\partial \xi^1} & \frac{\partial(\xi^1)^\dagger}{\partial \xi^2} \\ \frac{\partial(\xi^2)^\dagger}{\partial \xi^1} & \frac{\partial(\xi^2)^\dagger}{\partial \xi^2} \end{bmatrix}, \tag{15}$$

which we can use as follows to change our coordinates and metric:

$$\left(\xi^i\right)^\dagger = J\xi^i \tag{16}$$

$$g_{ij}^\dagger = J^T g_{ij} J, \tag{17}$$

where $(\, ^T \,)$ is the transpose.

To prove arclength invariance, it is sufficient to show that $(ds^2)^\dagger = ds^2$...

Let us now prove the invariance of area:

$$A = \iint\limits_M \sqrt{\det(g_{ij})} d\xi^1 d\xi^2. \tag{18}$$

We transform the integral into the $\left(\, ^\dagger \,\right)$ parametrization using the determinant of the Jacobian:

$$A = \iint\limits_M \sqrt{\det(g_{ij})} \det(J) \left(d\xi^1\right)^\dagger \left(d\xi^2\right)^\dagger = \iint\limits_M \sqrt{\det(J^T)\det(g_{ij})\det(J)} \left(d\xi^1\right)^\dagger \left(d\xi^2\right)^\dagger, \tag{19}$$

noting that $\det(J) = \det\left(J^T\right)$. Now simplify...

Question 2 - Eigenvalues of L

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$$\langle \bar{v}, Lv \rangle = \langle \bar{v}, G^{-1}Kv \rangle = \bar{v}^T G G^{-1} K v = \bar{v}^T K v,$$

with K symmetric.

$\therefore \quad Lv = \lambda v, \quad L\bar{v} = \bar{\lambda}\bar{v}, \quad v \not\equiv 0.$

$$\therefore \langle \bar{v}, Lv \rangle - \langle v, L\bar{v} \rangle = \lambda \langle \bar{v}, v \rangle - \bar{\lambda} \langle v, \bar{v} \rangle$$

$$= \lambda \bar{v}^T G v - \bar{\lambda} v^T G \bar{v}$$

$\therefore \quad \underbrace{\bar{v}^T K v - v^T K \bar{v}}_{\text{Zero as K symmetric}} = (\lambda - \bar{\lambda})\, \bar{v}^T G v$, noting G is symmetric

$\bar{v}^T G v$ → Not zero as G positive definite and $v \not\equiv 0$.

$\therefore \lambda = \bar{\lambda}$, eigvals real.

$\therefore$ v, eigenvectors real.

Orthogonality $\quad Lv_1 = \lambda_1 v_1 \quad Lv_2 = \lambda_2 v_2$

$$\langle v_2, Lv_1 \rangle - \langle v, Lv_2 \rangle = \lambda_1 v_2^T G v_1 - \lambda_2 v_1^T G v_2$$

$\therefore \underbrace{v_2^T K v_1 - v_1^T K v_2}_{\text{Zero as K symmetric}} = (\lambda_1 - \lambda_2)(v_2^T G v_1)$ as G symmetric

$$\therefore 0 = (\lambda_1 - \lambda_2) v_2^T G v_1 = \lambda_1 - \lambda_2 \langle v_2, v_1 \rangle$$

∴ Eigvectors associated with different eigvalues are orthogonal, w.r.t. this inner product, which is the appropriate one as

$\langle a, a \rangle = a^T G a$, the length of a vector

cf. $ds^2 = g_{ij} d\xi^i d\xi^j$

[redacted]

Question 3 - Euler's Theorem for Normal Curvature

There are a number of ways to show Euler's theorem for the normal curvature, however we will derive the result as follows:

The normal curvature is defined as:

$$k_n = -\boldsymbol{n} \cdot \frac{d\boldsymbol{t}}{ds}. \tag{20}$$

Using the orthogonality of $\boldsymbol{t}$ and $\boldsymbol{n}$, (i.e. expanding $0 = \frac{d}{ds}(\mathbf{n} \cdot \mathbf{t})$) we can write this as:

$$k_n = -\boldsymbol{n} \cdot \frac{d\boldsymbol{t}}{ds} = \boldsymbol{t} \cdot \frac{d\boldsymbol{n}}{ds} = \frac{d\boldsymbol{x} \cdot d\boldsymbol{n}}{ds^2}. \tag{21}$$

To proceed, recall that

$$d\mathbf{x} = \mathbf{r}_1 d\xi^1 + \mathbf{r}_2 d\xi^2. \tag{22}$$

If we take take a differential of

$$\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{||\mathbf{r}_1 \times \mathbf{r}_2||}$$

we will pick up a lot of terms, however any differential of the denominator will vanish when dotted with $d\mathbf{x}$, and differentials of the numerator, dotted with $d\mathbf{x}$, can then be shuffled using the vector triple product to obtain terms of the form

$$-\mathbf{n} \cdot \frac{\partial \mathbf{r}_i}{\partial \xi^j}$$

which are the components of the second fundamental form K. Putting it together, we get

$$k_n = -\frac{L\left(d\xi^1\right)^2 + 2M d\xi^1 d\xi^2 + N\left(d\xi^2\right)^2}{E\left(d\xi^1\right)^2 + 2F d\xi^1 d\xi^2 + G\left(d\xi^2\right)^2}, \tag{23}$$

where L, M, and N are the entries of the second fundamental form and E, F and G are the entries of the first fundamental form. In terms of the matrices K and G, we would write $L = K_{11}$, $M = K_{12}$, $N = K_{22}$, and $E = G_{11}$, $F = G_{12}$, $G = G_{22}$[1].

For simplicity, we consider a surface such that $M = F = 0$, so our normal curvature is now:

$$k_n = -\frac{L\left(d\xi^1\right)^2 + N\left(d\xi^2\right)^2}{E\left(d\xi^1\right)^2 + G\left(d\xi^2\right)^2}. \tag{24}$$

We can find the principal curvatures, k_1 and k_2, of this by considering two isoparametric curves, along one of which k_1 is constant and, along the other, k_2 is constant. We can choose a parametrization such that ξ^1 and ξ^2 are coordinates in the directions of these curves, so we can write our isoparametric curves as $\xi^1 = A$ and $\xi^2 = B$, where A and B are both constants.

As such, if we consider traveling along the isoparametric curve given by $\xi^1 = A$, then $d\xi^2 = 0$, while $\xi^2 = B$ and $d\xi^1 = 0$ on the other. This will yield explicit expressions for k_1 and k_2. Then, consider the angle θ between the tangent vector $\boldsymbol{\xi} = (\xi^1, \xi^2)$ and

[1] Forgive the unfortunate and embarrassing abuse of notation in doubly defining G – these are standard notations, though they rarely intersect like this!!

Let us now consider the angle between a general tangent vector $\boldsymbol{\xi} = (\xi^1, \xi^2)$ and the principal axis where k_1 is constant, which we can define as $\boldsymbol{u} = (u_1, 0)$ with a component only in the ξ^1 direction. Now take the inner product of these vectors using the generalized definitions of the dot product and arc-length that uses the first fundamental form...

Question 4 - The Monkey-Saddle

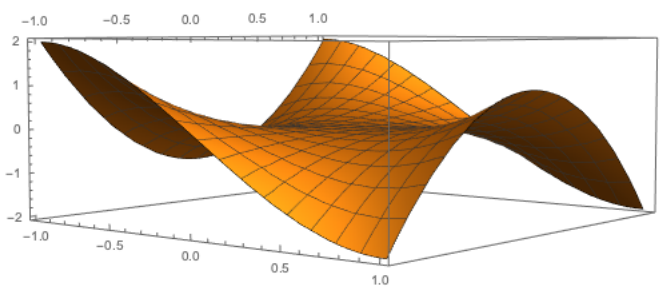
We are asked to find the principal, mean and Gaussian curvatures and to draw the Monkey Saddle surface, given by $z = x^3 - 3xy^2$, with the assistance of the Mathematica script "Curvature_computation.nb".

We can parametrize the surface as

$$\begin{aligned} x &= u & (25)\\ y &= v & (26)\\ z &= u^3 - 3uv^2, & (27) \end{aligned}$$

and it is a small change to the provided script to produce the surface and compute the curvatures.

The surface has the shape:



The Gaussian curvature, K_G, and mean curvature, H, are computed to be:

$$\begin{aligned} K_G &= -\frac{36\left(u^2+v^2\right)}{\left(1+9u^4+18u^2v^2+9v^4\right)^{2}} & (28)\\ H &= \frac{54\left(u^5-2u^3v^2-3uv^4\right)}{\left(1+9u^4+18u^2v^2+9v^4\right)^{\frac{3}{2}}}. & (29) \end{aligned}$$

Note that the denominator of K_G is positive for all real values of u and v, while the numerator is greater than 0 for all values of u and v except when $u = v = 0$, where it is equal to 0. This then implies that the Gaussian curvature is negative everywhere except at the origin, as required.

Question 5 - The Slightly Deformed Sphere

It is highly recommended that you use a symbolic algebra package like Mathematica or Maple to help you complete this problem; the algebra becomes messy very quickly. There are some similarities in computing the curvatures as in question 4, however, the main difference will be consistently working to first order in ϵ.

To begin with, we start with our position vector, defined as:

$$\boldsymbol{x} = R(1 + \epsilon h(\theta, \phi) \left\{\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)\right\}, \tag{30}$$

find the corresponding tangent vectors, $\boldsymbol{r}_\theta$ and $\boldsymbol{r}_\phi$:

$$\begin{aligned}\boldsymbol{r}_\theta = [&R\cos(\phi)\left(\epsilon\sin(\theta)h^{(1,0)}(\theta,\phi) + \epsilon\cos(\theta)h(\theta,\phi) + \cos(\theta)\right), \\ &R\sin(\phi)\left(\epsilon\sin(\theta)h^{(1,0)}(\theta,\phi) + \epsilon\cos(\theta)h(\theta,\phi) + \cos(\theta)\right), \\ &R\epsilon\cos(\theta)h^{(1,0)}(\theta,\phi) - R\sin(\theta)(\epsilon h(\theta,\phi) + 1)],\end{aligned} \tag{31}$$

$$\begin{aligned}\boldsymbol{r}_\phi = [&R\sin(\theta)\left(\epsilon\cos(\phi)h^{(0,1)}(\theta,\phi) - \sin(\phi)(\epsilon h(\theta,\phi) + 1)\right), \\ &R\sin(\theta)\left(\epsilon\sin(\phi)h^{(0,1)}(\theta,\phi) + \epsilon\cos(\phi)h(\theta,\phi) + \cos(\phi)\right), \\ &R\epsilon\cos(\theta)h^{(0,1)}(\theta,\phi)],\end{aligned} \tag{32}$$

and compute the unit normal as we did previously:

$$\boldsymbol{n} = \frac{\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi\|}. \tag{33}$$

We can now calculate the first fundamental form, G, however, we only work up to $O(\epsilon)$ and neglect higher ordered terms. This can be done in Mathematica by using the "Series" command to expand in powers of ϵ. To first order, we find:

$$G = \begin{pmatrix} R^2(2\epsilon h(\theta,\phi) + 1) & 0 \\ 0 & R^2(2\epsilon h(\theta,\phi) + 1)\sin^2(\theta) \end{pmatrix}. \tag{34}$$

Calculating the second fundamental form, K, again up to $O(\epsilon)$ only, we find that the entries of K are given by:

$$K_{11} = R\left(\epsilon h(\theta,\phi) - \epsilon h^{(2,0)}(\theta,\phi) + 1\right) \tag{35}$$

$$K_{12} = K_{21} = \frac{R\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{\sin(\theta)} \tag{36}$$

$$K_{22} = R\left(\epsilon h(\theta,\phi)\sin^2(\theta) + \left(\sin(\theta) - \epsilon\cos(\theta)h^{(1,0)}(\theta,\phi)\right)\sin(\theta) - \epsilon h^{(0,2)}(\theta,\phi)\right). \tag{37}$$

where $h^{(1,0)}(\theta,\phi) \equiv \frac{\partial h}{\partial\theta}$, $h^{(0,1)}(\theta,\phi) \equiv \frac{\partial h}{\partial\phi}$ and so forth.

Lastly, we obtain the entries of the principal curvature matrix, $L = G^{-1}K$ by combining the first order results. We again only take terms to $O(\epsilon)$:

$$L_{11} = \frac{1}{R} - \frac{\epsilon\left(h(\theta,\phi) + h^{(2,0)}(\theta,\phi)\right)}{R} \quad (38)$$

$$L_{12} = \frac{\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{R\sin(\theta)} \quad (39)$$

$$L_{21} = \frac{\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{R\sin^3(\theta)} \quad (40)$$

$$L_{22} = \frac{1}{R} - \frac{\epsilon\left(h^{(0,2)}(\theta,\phi)\csc^2(\theta) + h(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi)\right)}{R}. \quad (41)$$

Computing the Gaussian curvature, K_G, and mean curvature, H, to first order, we find:

$$H = \frac{\mathrm{tr}(L)}{2} = \frac{1}{R} - \frac{\epsilon\left(\left(h^{(2,0)}(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi) + \csc^2(\theta)h^{(0,2)}(\theta,\phi) + 2h(\theta,\phi)\right)\right)}{2R} \quad (42)$$

$$K_G = \det(L) = \frac{1}{R^2} - \frac{\epsilon\left(h^{(2,0)}(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi) + \csc^2(\theta)h^{(0,2)}(\theta,\phi) + 2h(\theta,\phi)\right)}{R^2}, \quad (43)$$

which are the required results.

A small Mathematica script which does this computation is provided below. Be sure to understand what the program and the commands actually do before just copying it!

Algorithm 1 A script to compute the Mean and Gaussian curvatures of a slightly deformed sphere to $O(\epsilon)$.

```
x = R (1 + ep*h[t, p])*{Cos[p] Sin[t], Sin[p] Sin[t], Cos[t]}
rt = Simplify[D[x, t]];
rp = Simplify[D[x, p]];
n = Simplify[Cross[rt, rp]/Sqrt[Cross[rt, rp].Cross[rt, rp]], Assumptions -> R > 0];
G = Simplify[{{rt.rt, rt.rp}, {rp.rt, rp.rp}}];
G1 = Simplify[Normal[Series[G, {ep, 0, 1}]]];
K = Simplify[{{-n.D[rt, t], -n.D[rt, p]}, {-n.D[rp, t], -n.D[rp, p]}}];
K1 = Simplify[Normal[Series[K, {ep, 0, 1}]]];
L1 = Normal[Series[Inverse[G1].K1, {ep, 0, 1}]];
GaussK1 = Series[Det[L1], {ep, 0, 1}]
H = 0.5*Simplify[Series[Tr[L1], {ep, 0, 1}]]
```