Problem Sheet 4: Solutions

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DUE ON FRIDAY OF WEEK 0 Trinity Term: questions 3 - 8 from Module III

Module III

Question 3 - Axon pulling.

The force balance reads

$$\frac{\partial \mathsf{n}_3}{\partial S} + \mathsf{f}_3 = 0 \tag{1}$$

and the constitutive relation is

$$\mathsf{n}_3 = EA(\alpha - 1) \tag{2}$$

where $\alpha = \partial s/\partial S$. Since $\mathsf{f}_3 = -k(s(S) - S)$, we can take an S derivative of (1) and use (2) to get

$$\frac{\partial^2 \mathsf{n}_3}{\partial S^2} + \frac{\partial \mathsf{f}_3}{\partial S} = \frac{\partial^2 \mathsf{n}_3}{\partial S^2} - k(\alpha - 1) = \frac{\partial^2 \mathsf{n}_3}{\partial S^2} - \frac{k}{EA}\mathsf{n}_3 = 0. \tag{3}$$

Due to the applied pulling tension, one boundary condition is

$$\mathsf{n}_3 = \sigma \text{ at } S = L.$$

At $S = 0$, $s = 0$ and thus $\mathsf{f}_3 = 0$, which implies the other boundary condition is

$$\frac{\partial \mathsf{n}_3}{\partial S} = 0 \text{ at } S = 0.$$

Solving, we get

$$\mathsf{n}_3 = \frac{\sigma \cosh(S/a)}{\cosh(L/a)}. \tag{4}$$

Note that the tension is an increasing function of arclength, taking maximum value at the pulling end $S = L$.

If the growth stretch γ is known, the evolution of a material point initially at position S_0 is given by

$$S(S_0, t) = \int_0^{S_0} \gamma dS_0. \tag{5}$$

Thus, for a general growth law of the form

$$\frac{\partial \gamma}{\partial t} = \gamma g(\mathsf{n}_3). \tag{6}$$

we can write

$$\begin{aligned} \partial_t S(S_0, t) &= \int_0^{S_0} (\partial_t \gamma) dS_0 \\ &= \int_0^{S_0} \gamma g(\mathsf{n}_3) dS_0 \\ &= \int_0^{S} g(\mathsf{n}_3(S)) dS. \end{aligned} \tag{7}$$

Therefore, taking $g = \hat{k}(\mathsf{n}_3 - \sigma^*)\mathrm{H}(\mathsf{n}_3 - \sigma^*)$ and inserting the explicit form (4) for the tension, the equation for the velocity is

$$\begin{aligned} V = \partial_t S &= \hat{k}\mathrm{H}(S - S^*) \int_{S^*}^{S} \left(\sigma_L \frac{\cosh(S/a)}{\cosh(L/a)} - \sigma^* \right) dS \\ &= \hat{k}\mathrm{H}(S - S^*) \left[\sigma^*(S^* - S) + a\sigma_L \frac{\sinh(S/a) - \sinh(S^*/a)}{\cosh(L/a)} \right], \end{aligned} \tag{8}$$

where $S^* = a \operatorname{arccosh}(\frac{\sigma^*}{\sigma_L} \cosh \frac{L}{a})$.

Question 4 - Gravitropism plus autotropism.

Similarly to the purely gravitropism case in the notes, the kinematics of the stem is described by

$$\frac{\partial x}{\partial s} = \sin\phi \tag{9}$$

$$\frac{\partial y}{\partial s} = \cos\phi \tag{10}$$

$$\frac{\partial \phi}{\partial s} = \mathsf{u}_2 \tag{11}$$

$$\frac{\partial \mathsf{u}_2}{\partial t} = -\beta\sin\phi - \nu\mathsf{u}_2 \tag{12}$$

In the limit of small ϕ (nearly vertical), we expand $\sin\phi \approx \phi$, $\cos\phi \approx 1$, which leads to the single equation for $x(s,t)$:

$$x_{sst} = -\beta x_s - \nu x_{ss}. \tag{13}$$

This can be integrated once with s:

$$x_{st} + \beta x + \nu x_s = c(t), \tag{14}$$

and the function $c(t)$ is determined from the boundary conditions

$$x(0) = 0, \phi(0) = \phi_0 \Rightarrow x_s(0,t) = \phi_0 \Rightarrow x_{st}(0,t) = 0$$

from which we conclude $c = \nu\phi_0$ is a constant.

In completing the system for x, we also have initial condition $x(s,0) = \phi_0 s$, which comes from an initially straight configuration, $\mathsf{u}_2(s,0) = 0$.

To solve the full nonlinear system, the simplest method is to integrate the system in space for a given curvature u_2, and then update the curvature by discretising (12), i.e.

$$\mathsf{u}_2(s, t+\Delta t) = \mathsf{u}_2(s,t) - \Delta t(\beta\sin\phi(s,t) - \nu\mathsf{u}_2(s,t)).$$

A sample Mathematica code, Tropism.nb is available on the course website.

Question 5 - Growing rod with diffusion.

The tricky aspect of this problem is being clear on the different configurations, in particular the fact that the diffusion takes place in the current configuration.
(a) We are given that the growth rate is proportional to growth stretch and nutrient concentration. This suggests

$$\frac{\partial \gamma}{\partial t} = K\gamma u$$

where $K > 0$ is a constant.
(b) The problem is symmetric with respect to the origin, so the solution for u is even and we only look at the solution for $s \geq 0$ (solutions for $s > 0$ or for both $s < 0$ and $s > 0$ are equally valid). Since diffusion is fast compared to growth, the nutrient is always in equilibrium, and thus satisfies

$$u_{ss} = Q/D,$$

which has solution

$$u = Qs^2 + C_1 s + C_2.$$

One boundary condition is $u = U$ at $s = l$. The other boundary condition depends on the total length. Since the concentration cannot go below 0, there is a critical length l^* beyond which the nutrient will be zero over some region near the middle. For $l < l^*$, the second constant is set by the behaviour at the origin where we have $u_s = 0$, that is $C_1 = 0$, which gives

$$u_1 = \frac{Q}{2D}(s^2 - l^2) + U.$$

The critical length is the value of l such that $u_1|_{(s=0)} = 0$, that is $l^* = \sqrt{2UD/Q}$, which defines the penetration length.

For $l > l^*$, setting $u = 0$ and $u_s = 0$ (since there can be no flux at the point where the concentration vanishes) at the point $s = a$ leads to

$$u_2 = \begin{cases} 0, & s < a, \\ \frac{Q}{2D}(s-a)^2 & s > a, \end{cases} \tag{15}$$

where a is determined from $u = U$ at $s = l$, giving

$$a = l - l^*.$$

Profiles for the nutrient concentration are shown in the Figure below.
(c) The current length is obtained from γ via

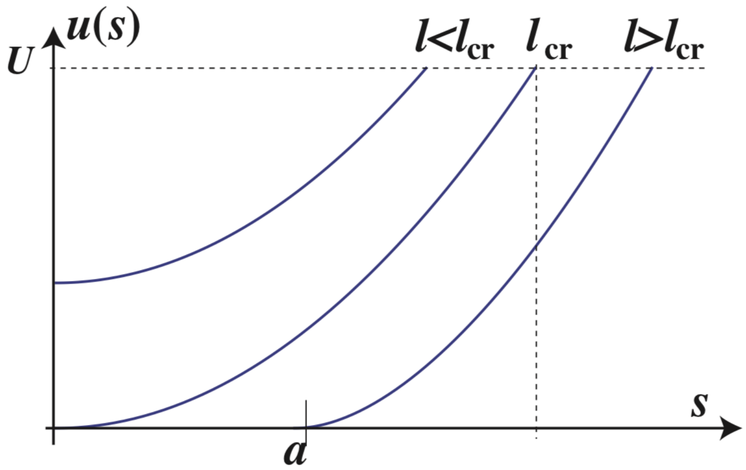
$$s = \int_0^s \gamma dS_0.$$

Thus, the length evolves according to

$$\partial_t s = \int_0^s \partial_t \gamma \, dS_0 = \int_0^s K\gamma u \, dS_0 = \int_0^s Ku(\sigma)\, d\sigma.$$

In particular, the length satisfies

$$\partial_t l = \int_0^l Ku(\sigma)\, d\sigma.$$



In the case $l < l^*$, we have

$$\partial_t l = K \int_0^l \frac{Q}{2D}(\sigma^2 - l^2) + U \, d\sigma$$

which can be integrated to give

$$\partial_t l = -\frac{KQ}{3D} l^3 + KUl.$$

In the asymptotic limit $l \ll l^*$, the l^3 term is small and we get the approximate solution

$$l(t) \sim L_0 \exp(KUt).$$

In the case $l > l^*$, we have

$$\partial_t l = \frac{KQ}{2D} \int_a^l (\sigma - a)^2 \, d\sigma,$$

from which we get

$$\partial_t l = \frac{KQ}{2D}(l-a)^3 = \frac{2KU}{3} l^*.$$

Thus,

$$l(t) = l^* + \frac{2KU}{3} l^*(t - t^*),$$

where t^* is the time at which $l = l^*$ (which we are not asked to work out here).

Question 6 - Buckling of an infinite planar rod on foundation.

The full system of equations is

$$\frac{\partial x}{\partial S} = \alpha\cos\theta, \qquad \frac{\partial y}{\partial S} = \alpha\sin\theta, \tag{16}$$

$$\frac{\partial F}{\partial S} + f = 0, \qquad \frac{\partial G}{\partial S} + g = 0, \tag{17}$$

$$EI\frac{\partial^2\theta}{\partial S^2} + \alpha G\cos\theta - \alpha F\sin\theta = 0. \tag{18}$$

Here, f and g are the x and y components of the foundation force

$$\mathbf{f} = \frac{h(\Delta)}{\gamma\Delta}\left[(x - S/\gamma)\mathbf{e}_x + (y - y_0)\mathbf{e}_y\right].$$

Taking $y_0 = 0$ and expanding $h(\Delta) \approx h(0) + h'(0)\Delta = -Ek\Delta$ (as in the lecture notes), we can write

$$f = -\frac{Ek}{\gamma}(x - S/\gamma), \quad g = -\frac{Ek}{\gamma}y.$$

These equations are supplemented by the constitutive law for tension

$$F\cos\theta + G\sin\theta = EA(\alpha - 1),$$

where A is the cross sectional area. We use this last relationship to express α in terms of F, G and θ in the equations above.

The first observation is that for $\gamma > 1$, there is a flat compressed solution, i.e. for which $\theta = y = 0$, and $\lambda = \alpha\gamma = 1$, which implies $\alpha = 1/\gamma$. This gives:

$$x^{(0)} = S/\gamma, \; y^{(0)} = \theta^{(0)} = G^{(0)} = 0, \; F^{(0)} = EA\frac{1-\gamma}{\gamma}. \tag{19}$$

To find the critical value of γ where a bifurcation first occurs, we expand our 5 variables in power series $x = x^{(0)} + \epsilon x^{(1)} + \mathrm{O}(\epsilon^2), y = y^{(0)} + \epsilon y^{(1)} + \mathrm{O}(\epsilon^2)$, etc.

Notice that α expands as

$$\alpha = \frac{1}{\gamma} + \epsilon\frac{1}{EA}F^{(1)} + \mathrm{O}(\epsilon^2).$$

Thus, since $\cos\theta = 1 + \mathrm{O}(\epsilon^2)$, and $\sin\theta = \epsilon\theta^{(1)} + \mathrm{O}(\epsilon^3)$, we get

$$\frac{dx^{(1)}}{dS} = \frac{F^{(1)}}{EA}, \quad \frac{dF^{(1)}}{dS} = \frac{Ek}{\gamma}x^{(1)},$$

which has solution $x^{(1)} = F^{(1)} = 0$. (One could also construct an exponential solution, but this is inconsistent with the physical problem, as could be seen by considering imposing boundary conditions on a finite domain, e.g. $x(0) = 0, x(L) = L$.)

We thus reduce the problem to a set of 3 linear equations

$$\frac{dy^{(1)}}{dS} = \frac{\theta^{(1)}}{\gamma}, \quad \frac{dG^{(1)}}{dS} = Ek\frac{y^{(1)}}{\gamma}, \tag{20}$$

$$\gamma^2 EI\frac{d^2\theta^{(1)}}{dS^2} + EA(\gamma - 1)\theta^{(1)} + \gamma G^{(1)} = 0. \tag{21}$$

And taking two derivates across the third equation leads to a single fourth order differential equation for $\theta^{(1)}$

$$\frac{d^4\theta^{(1)}}{dS^4} + 2a\frac{d^2\theta^{(1)}}{dS^2} + b^2\theta^{(1)} = 0, \tag{22}$$

where

$$a = \frac{A(\gamma-1)}{2I\gamma^2}, \quad b = \sqrt{\frac{k}{I\gamma^3}}, \tag{23}$$

We now look for solutions of the form $\theta^{(1)} \sim e^{i\omega S}$, which gives the quartic equation

$$\omega^4 + 2a\omega^2 + b^2 = 0$$

from which we obtain the 4 roots

$$\omega_1^2 = a + \sqrt{a^2-b^2}, \quad \omega_2^2 = a - \sqrt{a^2-b^2}. \tag{24}$$

For non-damped oscillations to exist we require $a \geq b$. The condition $a = b$ thus gives the first bifurcation condition for oscillatory modes on an infinite domain. Explicitly, it reads

$$A^2(\gamma-1)^2 - 4Ik\gamma = 0. \tag{25}$$

For a rod with circular cross section of radius r, we have $A = \pi r^2$, $I = \pi r^4/4$, and thus solving for γ gives

$$\gamma = 1 + \frac{k}{2\pi} + \frac{\sqrt{k(k+4\pi)}}{2\pi}.$$

Question 7 - The growing cuboid.

Since the block is isotropic and there is no load applied in the directions perpendicular to growth, we have $\alpha_1 = \alpha_2 =: \alpha$, which together with the incompressibility condition $\det \mathbf{A} = 1$ implies $\alpha_3 = 1/\alpha^2$. The rigid boundary condition along the $\mathbf{e}_3$-axis gives $\lambda_3 = 1$, from which we conclude $\alpha_3 = \gamma$, i.e. $\alpha = \gamma$.

The Cauchy equations for the Cauchy tensor $\mathbf{T} = \mathrm{diag}(t_1, t_2, t_3)$ are identically satisfied since the deformation is homogeneous. The constitutive equation, $\mathbf{T} = \mathbf{A}\partial W/\partial \mathbf{A} - p\mathbf{I}$ simplifies in component form to

$$t_1 = \alpha W_1(\alpha, \alpha, \alpha^{-2}) - p \tag{26}$$

$$t_2 = \alpha W_2(\alpha, \alpha, \alpha^{-2}) - p \tag{27}$$

$$t_3 = \alpha^{-2} W_3(\alpha, \alpha, \alpha^{-2}) - p \tag{28}$$

where $W_i = \partial_{\alpha_i} W$. Since t_1 and t_2 are constant and vanish at the boundary, these must be identically zero. Thus we can solve for the homeostatic pressure, giving

$$p = \alpha W_2(\alpha, \alpha, \alpha^{-2}).$$

Thus the stress generated during growth is

$$t_3 = \frac{1}{\gamma^2} W_3(\gamma, \gamma, \gamma^{-2}) - \gamma W_2(\gamma, \gamma, \gamma^{-2}).$$

(Here we have used the fact that $\alpha = \gamma$ in the deformation.) Now introduce the auxiliary function $\hat{W}(\gamma) = W(\gamma, \gamma, \gamma^{-2})$. Taking a derivative with chain rule gives

$$\hat{W}'(\gamma) = W_1 + W_2 - \frac{2}{\gamma^3} W_3.$$

Since $W_1 = W_2$ it follows that

$$\gamma \hat{W}'(\gamma) = 2(\gamma W_1 - \frac{2}{\gamma^2} W_3),$$

and we obtain

$$t_3 = -\frac{\gamma}{2} \hat{W}'(\gamma)$$

as desired.

Question 8 - Growing cylindrical tube.

The deformation gradient tensor, expressed in cylindrical coordinates, is $\mathbf{F} = \text{diag}(r'(R_0), r/R_0, \zeta)$. The elastic strain tensor is $\mathbf{A} = \text{diag}(\alpha_r, \alpha_\theta, \alpha_z)$, and the growth tensor for the symmetric growth is $\mathbf{G} = \text{diag}(\gamma_r, \gamma_\theta, \gamma_z)$. Here γ_r corresponds to radial growth, γ_θ to circumferential growth and γ_z to axial growth – each of these may be constant or functions of R_0, but we assume that they are input to the problem. This, along with the assumption that the tube remains cylindrical through the deformation, implies that the elastic stretches, i.e. components of $\mathbf{A}$, can also be functions of R_0, but not θ. Since $\mathbf{F} = \mathbf{AG}$, we have

$$\alpha_r = \frac{r'}{\gamma_r}, \quad \alpha_\theta = \frac{r}{R_0\gamma_\theta}, \quad \alpha_z = \frac{\zeta}{\gamma_z}. \tag{29}$$

Material incompressibility implies det$\mathbf{A}$=1, that is $\alpha_r\alpha_\theta\alpha_z = 1$ which implies

$$r dr = \frac{R_0 g(R_0)}{\zeta} dR_0, \tag{30}$$

that is

$$r^2 = a^2 + \frac{2}{\zeta}\int_{A_0}^{R_0} g(\rho)\rho d\rho \tag{31}$$

where $g(R_0) = \det \mathbf{G} = \gamma_r\gamma_\theta\gamma_z$.

The equilibrium conditions are given by the non-vanishing equation for the divergence of the Cauchy stress:

$$\frac{dt_r}{dr} + \frac{1}{r}(t_r - t_\theta) = 0, \tag{32}$$

and the application of the boundary conditions. The radial inflation pressure P gives the boundary condition

$$t_r = \begin{cases} 0, & r = b, \\ -P & r = a, \end{cases}$$

from which we obtain

$$\int_a^b \frac{t_\theta - t_r}{r} dr = P,$$

while the axial load condition is

$$2\pi \int_a^b t_z r\, dr\, d\theta = N.$$

The difficulty is that the bounds of the integrals a and b are not known – these must be determined through the deformation! It is therefore simpler to reformulate these integrals in the initial configuration. To do this we use (30) to convert to an integral over R_0. If we define

$$\tau(R_0; \zeta, a) := \int_{A_0}^{R_0} \frac{t_\theta - t_r}{\zeta\, r^2(\tilde{R}_0)}\, g(\tilde{R}_0)\tilde{R}_0 d\tilde{R}_0, \tag{33}$$

$$\phi(R_0; \zeta, a) := 2\pi \int_{A_0}^{R_0} \frac{t_z}{\zeta}\, g(\tilde{R}_0)\tilde{R}_0 d\tilde{R}_0, \tag{34}$$

then the two equations to find ζ and a as a function of P and N are

$$\tau(B_0; \zeta, a) = P, \tag{35}$$
$$\phi(B_0; \zeta, a) = N, \tag{36}$$

where $r(R_0)$ is given by (31).

We now turn to the constitutive law

$$t_r = \alpha_r \frac{\partial W}{\partial \alpha_r} - p, \quad t_\theta = \alpha_\theta \frac{\partial W}{\partial \alpha_\theta} - p, \quad t_z = \alpha_z \frac{\partial W}{\partial \alpha_z} - p. \tag{37}$$

These enable us to write the stress components in terms of the elastic components, since W is a given function of the α_i, and the α_i in turn can be expressed in terms of the unknowns a and ζ. The problem is they also contain the unknown hydrostatic pressure p, so the trick is to eliminate p from the integrals in τ and ϕ.

Considering τ first: clearly, we can write

$$t_\theta - t_r = \alpha_\theta \frac{\partial W}{\partial \alpha_\theta} - \alpha_r \frac{\partial W}{\partial \alpha_r};$$

then, combining this with the relationships

$$\alpha_r = \frac{R_0 \gamma_\theta \gamma_z}{r(R_0)\zeta}, \quad \alpha_\theta = \frac{r(R_0)}{R_0 \gamma_\theta}, \quad \alpha_z = \frac{\zeta}{\gamma_z}, \tag{38}$$

the integrand of (33) is well defined with the only unknowns being the constants a and ζ. (Keep in mind that we are imagining that the growth functions γ_i are given, therefore $r(R_0)$ is known except for the constants a and ζ.)

In terms of expressing t_z only in terms of a and ζ in the integrand for ϕ, one approach is to note that the radial stress is obtained as the indefinite integral

$$t_r(R_0) = -P + \tau(R_0; \zeta, a), \tag{39}$$

Since this expression does not involve p, we can then eliminate p between t_r and t_z, to give

$$t_z = t_r - \alpha_r \frac{\partial W}{\partial \alpha_r} + \alpha_z \frac{\partial W}{\partial \alpha_z}.$$

In principle, one could complete the calculation this way, but in practice it is very messy, as the integral for ϕ involves the indefinite integral $\tau(R_0; \zeta, a)$ in the integrand!

A much better approach is as follows: Due to the incompressibility, $\det \mathbf{A} = 1$, we can write

$$\alpha_r = \alpha_z^{-1} \alpha_\theta^{-1}.$$

Then, defining

$$\hat{W}(\alpha_\theta, \alpha_z) = W\left(\frac{1}{\alpha_z \alpha_\theta}, \alpha_\theta, \alpha_z\right),$$

we can compute

$$\frac{\partial \hat{W}}{\partial \alpha_\theta} = -\frac{1}{\alpha_\theta^2 \alpha_z} \frac{\partial W}{\partial \alpha_r} + \frac{\partial W}{\partial \alpha_\theta},$$

and thus

$$\alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta} = -\alpha_r \frac{\partial W}{\partial \alpha_r} + \alpha_\theta \frac{\partial W}{\partial \alpha_\theta}.$$

Therefore, we can write

$$t_\theta - t_r = \alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta}.$$

Similarly,

$$t_z - t_r = \alpha_z \frac{\partial \hat{W}}{\partial \alpha_z}.$$

Now, the big trick is to write

$$\int_a^b t_z r \, dr = \int_a^b r(t_z - t_r) + r t_r \, dr.$$

The first term can be replaced by $r\alpha_z \frac{\partial \hat{W}}{\partial \alpha_z}$, while we can integrate by parts on the second term and use (32):

$$\int_a^b r t_r \, dr = \frac{r^2}{2} t_r \Big|_a^b - \int_a^b \frac{r^2}{2} \frac{t_\theta - t_r}{r} \, dr = \frac{Pa^2}{2} - \frac{1}{2} \int_a^b r(t_\theta - t_r) \, dr.$$

Bringing it together, the axial load condition becomes

$$\pi \int_a^b (2\alpha_z \frac{\partial \hat{W}}{\partial \alpha_z} - \alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta}) r \, dr = N - Pa^2.$$

Converting to reference variables, the function ϕ is updated to

$$\phi(R_0, A_0; \zeta, a) = \frac{\pi}{\zeta} \int_{A_0}^{R_0} (2\alpha_z \frac{\partial \hat{W}}{\partial \alpha_z} - \alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta}) g(R_0) R_0 \, dR_0,$$

and the two conditions to solve for a and ζ are updated to

$$\tau(B_0; \zeta, a) = P, \tag{40}$$

$$\phi(B_0; \zeta, a) = N - Pa^2. \tag{41}$$