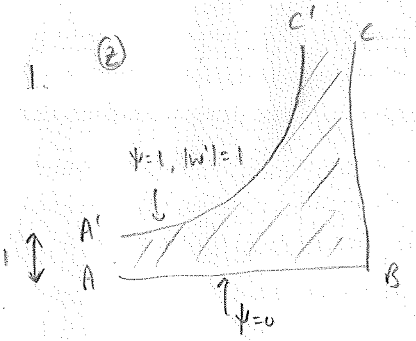
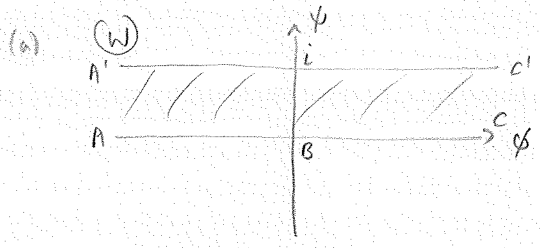
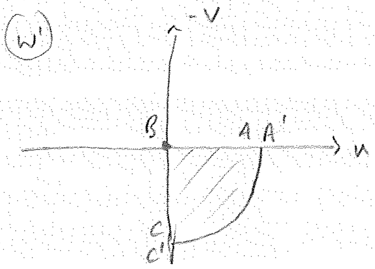
1.

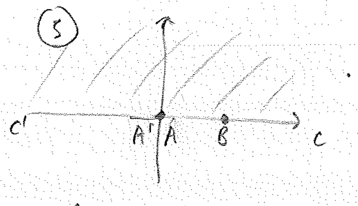


(a)

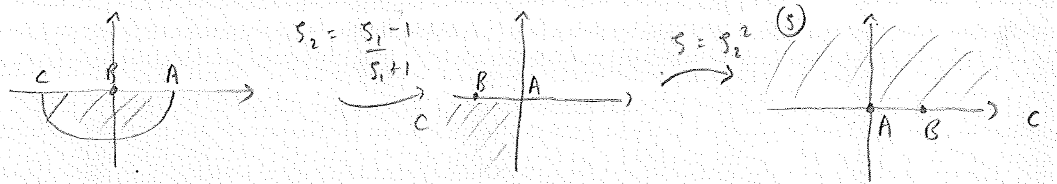




(b) $\zeta = e^{\pi W}$ maps W domain to



$\zeta_1 = W'^2$ maps W' domain to



so $\zeta = \left(\frac{W'^2-1}{W'^2+1}\right)^2$ maps W' domain onto the upper half ζ plane.

Hence $\boxed{e^{\pi W} = \left(\frac{W'^2-1}{W'^2+1}\right)^2}$ $\Rightarrow$ $W'^2\left(e^{\frac{\pi W}{2}}-1\right) = -\left(e^{\frac{\pi W}{2}}+1\right)$ $\Rightarrow$ $W'^2 = \frac{1+e^{\pi W/2}}{1-e^{\pi W/2}} = -\coth\frac{\pi W}{4}$.

(c) If $W' = e^{-i\theta}$ on free surface (θ goes from 0 at A' to $\frac{\pi}{2}$ at C')

then $\zeta = \left(\frac{e^{-2i\theta}-1}{e^{-2i\theta}+1}\right)^2 = \left(\frac{-i\sin\theta}{\cos\theta}\right)^2 = -\tan^2\theta = e^{\pi W}$ $\left(\Rightarrow -\frac{2\sin\theta}{\cos^3\theta} = \pi e^{\pi W}\frac{dW}{d\theta} \Rightarrow \frac{dW}{d\theta} = \frac{2}{\pi\sin\theta\cos\theta}\right)$

Also $\frac{dz}{d\theta} = \frac{1}{W'}\frac{dW}{d\theta} = e^{i\theta}\cdot\frac{2}{\pi\sin\theta\cos\theta}$

$$= \frac{2}{\pi}\left(\frac{1}{\sin\theta} + \frac{i}{\cos\theta}\right)$$

(d) Taking real and imaginary parts $\frac{dx}{d\theta} = \frac{2}{\pi\sin\theta}$ $\frac{dy}{d\theta} = \frac{2}{\pi\cos\theta}$

[If $t = \tan\frac{1}{2}\theta$ then $\tan\theta = \frac{2t}{1-t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, and $dt = \frac{1}{2}\sec^2\frac{1}{2}\theta\, d\theta = \frac{1+t^2}{2}d\theta$

so $\int\frac{d\theta}{\sin\theta} = \int\frac{1+t^2}{2t}\frac{2dt}{1+t^2} = \int\frac{dt}{t} = \log(\tan\frac{1}{2}\theta)$ & $\int\frac{d\theta}{\cos\theta} = \int\frac{d(\theta+\frac{\pi}{2})}{\sin(\theta+\frac{\pi}{2})} = \log\left(\tan\left(\frac{1}{2}\theta+\frac{\pi}{4}\right)\right)$

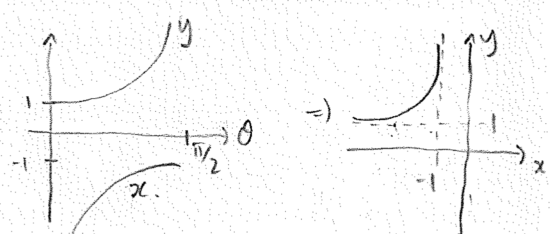
(or) $= \int\frac{2dt}{1-t^2} = 2\tanh^{-1}(\tan\frac{1}{2}\theta)$]

so $x = x_0 + \frac{2}{\pi}\log(\tan\frac{1}{2}\theta)$ $y = y_0 + \frac{2}{\pi}\log\left(\tan\left(\frac{1}{2}\theta+\frac{\pi}{4}\right)\right)$

As $x\to-\infty$, $\theta\to 0$, $y\to 1$ $\Rightarrow$ $y_0 = 1$

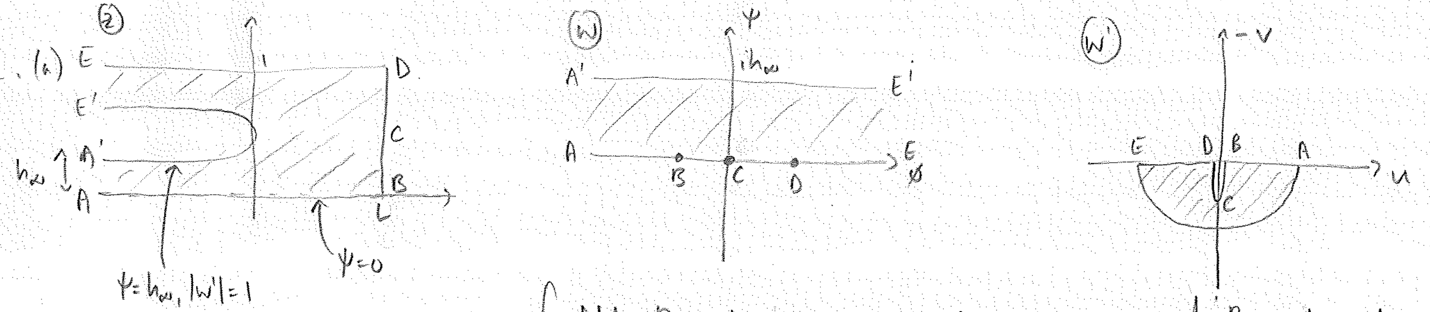
As $y\to\infty$, $\theta\to\frac{\pi}{2}$, $x\to -1$ $\Rightarrow$ $x_0 = -1$

so $\boxed{x = -1 + \frac{2}{\pi}\log(\tan\frac{1}{2}\theta) \quad y = 1 + \frac{2}{\pi}\underbrace{\log\left(\tan\left(\frac{1}{2}\theta+\frac{\pi}{4}\right)\right)}_{2\tanh^{-1}(\tan\frac{1}{2}\theta)}}$



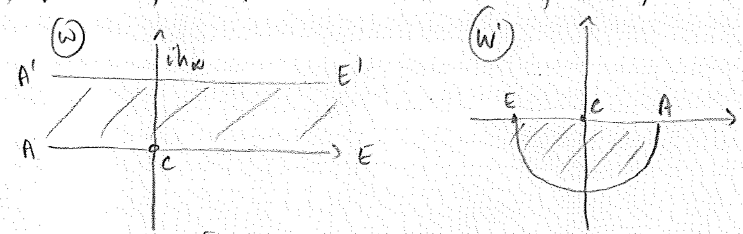
Eliminating $\tan\frac{\theta}{2}$ $\Rightarrow$ $e^{\frac{\pi}{2}(x+1)} = \tanh\frac{\pi}{4}(y-1)$ (there are a number of ways of writing this)

2. (a)



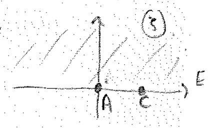
[Note the values of ϕ at B, D are unknown, & the value of v at C is unknown.]

(b) If $L \to \infty$, the fluid has zero velocity at ∞, so BCD coincide in both w and w' planes.



$\zeta = e^{\pi w / h_\infty}$ maps w domain to Uhp

$\zeta = \left(\dfrac{w'-1}{w'+1}\right)^2$ maps w' domain to Uhp with same image pts.



So $\zeta = e^{\frac{\pi w}{h_\infty}} = \left(\dfrac{w'-1}{w'+1}\right)^2 \Rightarrow e^{\frac{\pi w}{2h_\infty}} = \dfrac{1-w'}{1+w'}$

(note $w' = 0$ when $w = 0$ at C for correct choice of square root)

$$\Rightarrow \boxed{w' = \frac{1 - e^{\frac{\pi w}{2h_\infty}}}{1 + e^{\frac{\pi w}{2h_\infty}}} = -\tanh\frac{\pi w}{4h_\infty}}$$

$$\int \frac{dw}{\tanh\frac{\pi w}{4h_\infty}} = -\int dz \Rightarrow \frac{4h_\infty}{\pi}\log\left(\sinh\frac{\pi w}{4h_\infty}\right) = -\left(z - \frac{i}{2}\right) + \text{const}$$

$\sinh\frac{\pi i}{4} = i\sin\frac{\pi}{4} = \frac{i}{\sqrt{2}}$

$$= -\left(z - \frac{i}{2}\right) + \frac{4h_\infty}{\pi}\log\left(\frac{i}{\sqrt{2}}\right)$$ (since $w = ih_\infty$ at $z = \frac{i}{2}$)

$$\Rightarrow \boxed{\sinh\frac{\pi w}{4h_\infty} = \frac{i}{\sqrt{2}} e^{-\frac{\pi}{4h_\infty}\left(z - \frac{i}{2}\right)}}$$

On the free surface, $w = \phi + ih_\infty$, so $\sinh\frac{\pi w}{4h_\infty} = \sinh\frac{\pi\phi}{4h_\infty}\underbrace{\cos\frac{\pi}{4}}_{1/\sqrt{2}} + i\cosh\frac{\pi\phi}{4h_\infty}\underbrace{\sin\frac{\pi}{4}}_{1/\sqrt{2}}$

$$\Rightarrow \sinh\frac{\pi\phi}{4h_\infty} + i\cosh\frac{\pi\phi}{4h_\infty} = e^{-\frac{\pi}{4h_\infty}x}\left(i\cos\frac{\pi}{4h_\infty}\left(y - \frac{1}{2}\right) + \sin\frac{\pi}{4h_\infty}\left(y - \frac{1}{2}\right)\right)$$

comparing real & imaginary parts $1 = \cosh^2\frac{\pi\phi}{4h_\infty} - \sinh^2\frac{\pi\phi}{4h_\infty} = e^{-\frac{\pi x}{2h_\infty}}\left(\cos^2\frac{\pi}{4h_\infty}\left(y - \frac{1}{2}\right) - \sin^2\frac{\pi}{4h_\infty}\left(y - \frac{1}{2}\right)\right)$

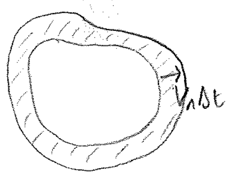
$$\Rightarrow \boxed{1 = e^{-\frac{\pi x}{2h_\infty}}\cos\frac{\pi}{2h_\infty}\left(y - \frac{1}{2}\right)}$$

As $x \to -\infty$, $y \to h_\infty$ and $1 - h_\infty$, so need $\cos\frac{\pi}{2h_\infty}\left(h_\infty - \frac{1}{2}\right) = 0 \Rightarrow \frac{\pi}{2h_\infty}\left(h_\infty - \frac{1}{2}\right) = \pm\frac{\pi}{2} \Rightarrow \boxed{h_\infty = \frac{1}{4}}$

[Alternatively, $\sinh^2\frac{\pi w}{4h_\infty} = \frac{1}{2}\left(\cosh\frac{\pi w}{2h_\infty} - 1\right) = -\frac{1}{2}e^{-\frac{\pi}{2h_\infty}\left(z - \frac{i}{2}\right)}$ & $\cosh\frac{\pi w}{2h_\infty} = i\sinh\frac{\pi\phi}{2h_\infty}$ on free surface $w = \phi + ih_\infty$ so taking real parts immediately gives $1 = e^{-\frac{\pi x}{2h_\infty}}\cos\frac{\pi}{2h_\infty}\left(y - \frac{1}{2}\right)$]

3. (a) $M_n(t) = \iint_{D(t)} z^n \, dx\,dy$: $M_0(t)$ = area

$M_1(t)$ = centre of mass × area. (i.e. $\frac{M_1(t)}{M_0(t)}$ = centre of mass).



$$M_n(t+\Delta t) - M_n(t) = \int_{\partial D(t)} z^n V_n \Delta t \, ds$$ (i.e. shaded region in diagram).

so $\frac{M_n(t+\Delta t) - M_n(t)}{\Delta t} = \int_{\partial D} z^n V_n \, ds$ and taking $\Delta t \to 0$ gives $\boxed{\frac{dM_n}{dt} = \int_{\partial D} z^n V_n \, ds}$

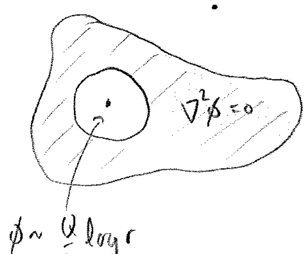
[Reynold's Transport Theorem gives & divergence theorem $\frac{d}{dt}\iint_{D(t)} z^n dx\,dy = \iint_D \nabla\cdot(z^n \underline{u})\, dx\,dy = \int_{\partial D} z^n \underline{u}\cdot\underline{n}\, ds = \int_{\partial D} z^n V_n \, ds$]

(b) $\frac{\partial G}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial G}{\partial x} + i\frac{\partial G}{\partial y}\right)$ so $\iint_R \frac{\partial G}{\partial \bar z}\, dx\,dy = \frac{1}{2}\iint_R \frac{\partial G}{\partial x} + i\frac{\partial G}{\partial y}\, dx\,dy = \frac{1}{2}\int_{\partial R} G\,dy - iG\,dx$

$$= \frac{1}{2i}\int_{\partial R} G(dx + i\,dy) = \frac{1}{2i}\int_{\partial R} G\, dz.$$

If $G = z^n \bar z$ then $\frac{\partial G}{\partial \bar z} = z^n$, so $M_n(t) = \iint_{D(t)} \frac{\partial G}{\partial \bar z}\, dx\,dy = \frac{1}{2i}\int_{\partial D(t)} G\, dz = \frac{1}{2i}\int_{\partial D(t)} z^n \bar z \, dz.$

(c)



Consider $$\iint_{R = D(t)\setminus D(0,\varepsilon)} \phi \underbrace{\cancel{\nabla^2(z^n)}}_{0} - (z^n)\underbrace{\cancel{\nabla^2\phi}}_{0}\, dx\,dy = \int_{\partial R} \phi \frac{\partial}{\partial n}(z^n) - (z^n)\frac{\partial \phi}{\partial n}\, ds$$

(using Green's second identity with $u = \phi$, $v = z^n$)

$$\Rightarrow \quad 0 = \underbrace{\int_{\partial D(t)} \phi \underbrace{\frac{\partial}{\partial n}(z^n)}_{0} - \underbrace{(z^n)\frac{\partial \phi}{\partial n}}_{z^n V_n} ds}_{-\frac{dM_n}{dt} \text{ from (a)}} - \underbrace{\int_{\partial D(0,\varepsilon)} \phi \frac{\partial}{\partial n}(z^n) - (z^n)\frac{\partial\phi}{\partial n}\, ds}$$

write $z = re^{i\theta}$ $\theta \in (0, 2\pi)$.

$\phi \sim \frac{Q}{2\pi}\log r$, $\frac{\partial \phi}{\partial n} \sim \frac{Q}{2\pi r}$

$z^n = r^n e^{in\theta}$ $\frac{\partial}{\partial n}(z^n) = n r^{n-1} e^{in\theta}$

$$\Rightarrow \frac{dM_n}{dt} \sim -\int_{0,\ r=\varepsilon}^{2\pi} \left(\frac{Q}{2\pi}\log\varepsilon \cdot n\varepsilon^{n-1} e^{in\theta} - \frac{Q}{2\pi}\varepsilon^{n-1} e^{in\theta}\right)\varepsilon\, d\theta \begin{cases} \to 0 \text{ as } \varepsilon \to 0 & \text{for } n > 0. \\ = Q & \text{for } n = 0. \end{cases}$$

(d) $z = F(\varsigma, t)$ maps $|\varsigma|<1$ to $D(t)$ with $F(0,t)=0$.

$$M_n(t) = \frac{1}{2i}\int_{\partial D(t)} z^n \bar{z}\, dz \quad \text{from (b)}$$

$$dz = \frac{\partial F}{\partial \varsigma} d\varsigma.$$

$$= \frac{1}{2i}\int_{|\varsigma|=1} F(\varsigma,t)^n \overline{F(\varsigma,t)} \frac{\partial F}{\partial \varsigma}(\varsigma,t)\, d\varsigma.$$

If $F(\varsigma,t) = \sum_{j=1}^{m} a_j(t)\varsigma^j$, then $M_n = \frac{1}{2i}\int_{|\varsigma|=1} \underbrace{\left(\sum_1^m a_j \varsigma^j\right)^n \left(\sum_1^m \bar{a}_j \varsigma^{-j}\right)\left(\sum_1^m a_j j \varsigma^{j-1}\right)}_{\text{lowest power of } \varsigma \text{ is } a_1^{n+1}\bar{a}_m \varsigma^{n-m}} d\varsigma.$

The integrand is holomorphic in $|\varsigma|\le 1$ if $n \geq m$, so Cauchy's Theorem $\Rightarrow$ $\boxed{M_n = 0 \text{ for } n \geq m}$

For $0 \le n < m$, M_n is given by Cauchy's residue as $\pi \times c_{-1}$ where c_{-1} is the coefficient of ς^{-1}.

eg. for $n=0$ this is $\boxed{\pi \sum_{j=1}^{m} j|a_j|^2}$ (ie must take the same j in each sum to get ς^{-1}).

for $n = m-1$ it is $\boxed{\pi a_1^m \bar{a}_m}$ (ie. must take the lowest power in each sum to get ς^{-1})

If $m=2$, $M_0 = \frac{1}{2i}\int_{|\varsigma|=1} \underbrace{(\bar{a}_1\varsigma^{-1} + \bar{a}_2\varsigma^{-2})(a_1 + 2a_2\varsigma)}_{(\bar{a}_1 a_1 + 2\bar{a}_2 a_2)\varsigma^{-1}} d\varsigma = \boxed{\pi(|a_1|^2 + 2|a_2|^2).}$

$M_1 = \frac{1}{2i}\int_{|\varsigma|=1} \underbrace{(a_1\varsigma + a_2\varsigma^2)(\bar{a}_1\varsigma^{-1} + \bar{a}_2\varsigma^{-2})(a_1 + 2a_2\varsigma)}_{a_1^2\bar{a}_2\varsigma^{-1}} d\varsigma = \boxed{\pi a_1^2 \bar{a}_2}$

Combining with $\frac{dM_0}{dt} = Q$, $\frac{dM_1}{dt} = 0$, we have $\pi(|a_1(t)|^2 + 2|a_2(t)|^2) = Qt + \pi(|a_1(0)|^2 + 2|a_2(0)|^2)$

$$\pi a_1(t)^2 \overline{a_2(t)} = \pi a_1(0)^2 \overline{a_2(0)}$$

From above, we have general formulae. $\boxed{M_0 = \pi \sum_{j=1}^{m} j|a_j|^2}$ and $\boxed{M_{m-1} = \pi a_1^m \bar{a}_m}$