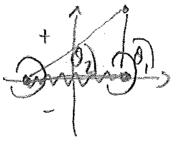
1. $w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)\,ds}{s-z}$ For t on Γ, $\boxed{w_\pm(t) = \pm\frac{1}{2} f(t) + \frac{1}{2\pi i} \fint_\Gamma \frac{f(s)\,ds}{s-t}}$

($\fint_\Gamma \frac{ds}{s-t}$ is the principal value integral, i.e. $\lim_{\varepsilon\to 0}\left(\int_a^{t-\varepsilon}\frac{ds}{s-t} + \int_{t+\varepsilon}^{b}\frac{ds}{s-t}\right)$)

• Let $W(z) = \frac{(z-1)^{\alpha-1}}{(z+1)^{\alpha}} = \frac{|z-1|^{\alpha-1}}{|z+1|^{\alpha}} e^{i(\theta_1-\theta_2)\alpha - i\theta_1}$ where $\theta_1 = \arg(z-1)\in(-\pi,\pi]$ & $\theta_2 = \arg(z+1)\in(-\pi,\pi]$



Branch cut on $[-1,1]$. Above cut, $\theta_1=\pi, \theta_2=0$ $\Rightarrow$ $W_+(x) = -e^{i\pi\alpha}\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}$

Below cut, $\theta_1=-\pi, \theta_2=0$ $\Rightarrow$ $W_-(x) = -e^{-i\pi\alpha}\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}$

If we also write $W(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{f(t)\,dt}{t-z}$ as a Cauchy integral, Plemelj tells us that

$$f(x) = W_+(x) - W_-(x) = -2i\sin\pi\alpha\,\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}$$

$$\& \quad \underbrace{\frac{1}{\pi i}\fint_{-1}^{1}\frac{f(t)\,dt}{t-x}}_{-\frac{2}{\pi}\sin\pi\alpha\fint_{-1}^{1}\frac{(1-t)^{\alpha-1}}{(1+t)^{\alpha}(t-x)}dt} = W_+(x) + W_-(x) = -2\cos\pi\alpha\,\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}$$

$$\Rightarrow \boxed{\fint_{-1}^{1}\frac{(1-t)^{\alpha-1}\,dt}{(1+t)^{\alpha}(t-x)} = \frac{\pi}{\tan\pi\alpha}\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}}$$

2. (a) Look for a solution of form $w(z) = \frac{1}{2\pi i}\int_0^c \frac{f(\xi)}{\xi - z}\,d\xi$ for some ξ.

- Plemelj $\Rightarrow$ $w_+ + w_- = \frac{1}{\pi i}⨍_0^c \frac{f(\xi)}{\xi - x}\,d\xi$

  $w_+ - w_- = f(x)$.

- If $\mathrm{Im}(w_+ + w_-) = g_+ + g_- = 0$ then f must be pure imaginary, and hence $w_+ - w_- = i\,\mathrm{Im}(w_+ - w_-) = 2ig_+$

  $\Rightarrow$ $f(x) = 2ig_+(x)$.

  so $w(z) = \frac{1}{\pi}\int_0^c \frac{g_+(\xi)}{\xi - z}\,d\xi + h(z)$ where $h(z)$ is arbitrary function holomorphic except at $\{0,c\}$ and having $\mathrm{Im}(h_\pm) = 0$ on $[0,c]$. (i.e. a soln of homogeneous problem) (*)

- If w has at most inverse square root singularities at $0, c$, then $h(z)$ must have removable singularities there (i.e. be holomorphic there), and is therefore entire. If in addition $w \to 0$ as $z \to \infty$, then $h \to 0$ as $z \to \infty$, so by Liouville's theorem $h = 0$.

[(*) It is not obvious this is the only solution to the homogeneous problem, and in fact it is not, (eg. $z^{1/2}(c-z)^{1/2}$ is another.) Luckily the question only says to show this is <u>a</u> solution for $w(z)$.]

(b) As above, but now if $g_+ = g_- = g(x)$ then $\mathrm{Im}(w_+ - w_-) = 0 \Rightarrow f$ must be purely real, hence

$$w_+ + w_- = i\,\mathrm{Im}(w_+ + w_-) = 2ig(x) \Rightarrow \boxed{-2g(x) = ⨍_0^c \frac{f(\xi)\,d\xi}{\xi - x}} \quad \text{S.I.E for } f(x).$$

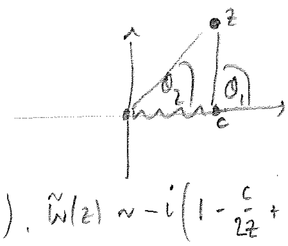
If $\tilde{w}(z)$ is holomorphic on $\mathbb{C}\setminus[0,c]$ and has $\tilde{w}_+ + \tilde{w}_- = 0$ on $(0,c)$, then let $\boxed{W(z) = \frac{w(z)}{\tilde{w}(z)}}$

$W_+ - W_- = \frac{w_+ + w_-}{\tilde{w}_+} = \frac{2ig(x)}{\tilde{w}_+(x)}$, so writing $W(z) = \frac{1}{2\pi i}\int_0^c \frac{F(\xi)}{\xi - z}\,d\xi$, we can read off from the Plemelj formulae that $F(x) = \frac{2ig(x)}{\tilde{w}_+(x)}$ (since $W_+ + W_- = \frac{1}{\pi i}⨍_0^c \frac{F(\xi)}{\xi - x}\,d\xi$, $W_+ - W_- = F(x)$).

Hence $W(z) = \frac{1}{\pi}\int_0^c \frac{g(\xi)}{\tilde{w}_+(\xi)(\xi - z)}\,d\xi + H(z)$, adding arbitrary $H(z)$, holomorphic on $\mathbb{C}\setminus\{0,c\}$.

i.e. $$\boxed{w(z) = \frac{\tilde{w}(z)}{\pi}\int_0^c \frac{g(\xi)}{\tilde{w}_+(\xi)(\xi - z)}\,d\xi + H(z)\tilde{w}(z)}$$

(c) Take $\tilde{w}(z) = \left(\frac{c-z}{z}\right)^{1/2} = \frac{|c-z|^{1/2}}{|z|^{1/2}}e^{i(\theta_1 - \theta_2 - \pi)}$ with $\theta_1, \theta_2 \in (-\pi, \pi]$



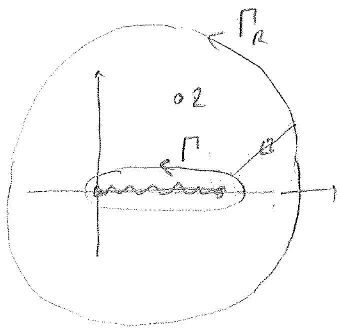
so on top of cut $(\theta_1 = \pi, \theta_2 = 0)$ $\tilde{w}_+(x) = \left(\frac{c-x}{x}\right)^{1/2}$,

bottom $(\theta_1 = -\pi, \theta_2 = 0)$ $\tilde{w}_-(x) = -\left(\frac{c-x}{x}\right)^{1/2}$, and at ∞, $(\theta_1 \sim \theta_2)$, $\tilde{w}(z) \sim -i\left(1 - \frac{c}{2z} + \dots\right)$

From (b), using this $\tilde{w}(z)$ and $g(x) = -\alpha$, $w(z) = -\frac{\alpha}{\pi}\left(\frac{c-z}{z}\right)^{1/2}\int_0^c \frac{\xi^{1/2}\,d\xi}{(c-\xi)^{1/2}(\xi - z)} + \left(\frac{c-z}{z}\right)^{1/2}H(z)$.

$H(z)$ cannot have a pole at 0 or c, else the singular behaviour of $w(z)$ would be too severe. So $H(z)$ is entire, and since $H \to 0$ as $z \to \infty$ (in order for $w \to 0$), Liouville's theorem implies <u>$H = 0$</u>.

(d) We have $w(z) = -\frac{\alpha}{\pi}\left(\frac{c-z}{z}\right)^{1/2}\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}(\xi-z)}\,d\xi$.



To calculate the integral term, consider $f(s)=\frac{1}{\tilde{w}(s)(s-z)}$, and

$$\int_{\Gamma_R} f(s)\,ds - \int_{\Gamma} f(s)\,ds = 2\pi i \,\mathrm{res}(f,z) \qquad \text{(C.R.T.)}$$

Note
- $\int_{\Gamma} f(s)\,ds = \underline{-2\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}(\xi-z)}\,d\xi}$.
- $\int_{\Gamma_R} f(s)\,ds \sim \int \frac{ds}{\underbrace{-i\left(1-\frac{c}{2s}+\dots\right)}_{\text{carefully evaluating } \tilde{w}(s) \text{ at large } s.}s\left(1-\frac{z}{s}\right)} \underset{s=Re^{i\theta}}{\sim} \int_0^{2\pi} \frac{Rie^{i\theta}}{-iRe^{i\theta}} + O\left(\frac{1}{R}\right) = \underline{-2\pi}$ as $R\to\infty$.
- $\mathrm{res}(f,z) = \left(\frac{z}{c-z}\right)^{1/2}$

So $2\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}(\xi-z)}\,d\xi = 2\pi i\left(\frac{z}{c-z}\right)^{1/2} - (-2\pi) \Rightarrow \int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}(\xi-z)}\,d\xi = \pi + \pi i\left(\frac{z}{c-z}\right)^{1/2}$.

Hence $\boxed{w(z) = -i\alpha - \alpha\left(\frac{c-z}{z}\right)^{1/2}}$

At large z, $w \sim -i\alpha + \alpha i\left(1-\frac{c}{2z}+\dots\right) \sim -\frac{i\alpha c}{2z} + \dots$

so if we know $w \sim \frac{i\hat{\Gamma}}{2\pi z}$, we can read off that $\boxed{\hat{\Gamma} = -\pi c\alpha.}$

3. (a) $a(t) f(t) + \dfrac{b(t)}{\pi i} \displaystyle\int_\Gamma \frac{f(s)\,ds}{s-t} = c(t)$ on Γ

If $W(z) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f(s)\,ds}{s-z}$ then Plemelj ⇒ $W_+ + W_- = \dfrac{1}{\pi i}\displaystyle\int_\Gamma \frac{f(s)\,ds}{s-t}$ and $W_+ - W_- = f(t)$, so

the above becomes $\quad a(t)(W_+ - W_-) + b(t)(W_+ + W_-) = c(t)$

i.e. $\quad (a+b)W_+ + (b-a)W_- = c$ on Γ.

(b) If $\tilde W(z)$ satisfies $(a+b)\tilde W_+ + (b-a)\tilde W_- = 0$ then $\dfrac{W_-}{\tilde W_-} = -\dfrac{b-a}{b+a}\dfrac{W_-}{\tilde W_+}$, so

$$\left(\frac{W}{\tilde W}\right)_+ - \left(\frac{W}{\tilde W}\right)_- = \frac{W_+ + \left(\frac{b-a}{b+a}\right)W_-}{\tilde W_+} = \frac{c}{\tilde W_+(a+b)} \quad \text{on } \Gamma.$$

(c) From Plemelj, $\mathcal{W} = \dfrac{W}{\tilde W} = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{F(s)}{s-z}\,ds$, we can read off that $\boxed{F(s) = \dfrac{c(s)}{\tilde W_+(s)(a(s)+b(s))}}$,

and hence a possible solution is $\boxed{W(z) = \dfrac{\tilde W(z)}{2\pi i}\displaystyle\int_\Gamma \frac{c(s)}{\tilde W_+(s)(a(s)+b(s))(s-z)}\,ds.}$

Moreover, $W_+ - W_- = f(t)$ (from Plemelj formula for W), and $\tilde W_- = -\left(\dfrac{b+a}{b-a}\right)\tilde W_+$, so

$$W_+ - W_- = \mathcal{W}_+\tilde W_+ - \mathcal{W}_-\tilde W_- = \left(\mathcal{W}_+ + \frac{b+a}{b-a}\mathcal{W}_-\right)\tilde W_+ = \frac{b(\mathcal{W}_+ + \mathcal{W}_-) - a(\mathcal{W}_+ - \mathcal{W}_-)}{b-a}\tilde W_+$$

$$= \frac{b\tilde W_+}{b-a}\frac{1}{\pi i}\int_\Gamma \frac{F(s)}{s-t}\,ds - \frac{a\tilde W_+}{b-a}F(t)$$

i.e. $\boxed{f(t) = \dfrac{b(t)\tilde W_+(t)}{b(t)-a(t)}\dfrac{1}{\pi i}\displaystyle\int_\Gamma \frac{c(s)}{(a(s)+b(s))\tilde W_+(s)(s-t)}\,ds - \frac{a(t)\,c(t)}{b(t)^2-c(t)^2}}$