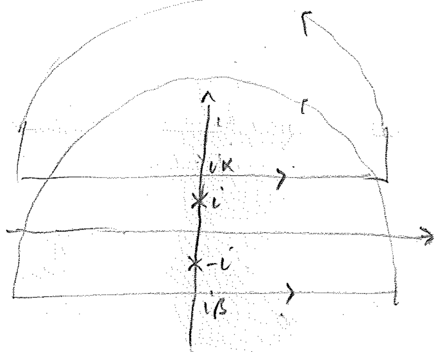
1. (a) $f_+(x) = \begin{cases} 0 & x<0 \\ e^{x} & x>0 \end{cases}$ $\qquad f_-(x) = \begin{cases} e^{-x} & x<0 \\ 0 & x>0 \end{cases}$

$$\bar{f}_+(k) = \int_0^\infty e^{x+ikx}\,dx = \left.\frac{e^{x(1+ik)}}{1+ik}\right|_0^\infty = -\frac{1}{1+ik} = \frac{+i}{k-i} \quad \text{for } \operatorname{Im} k > 1.$$

$$\bar{f}_-(k) = \int_{-\infty}^0 e^{-x+ikx}\,dx = \left.\frac{e^{x(-1+ik)}}{-1+ik}\right|_{-\infty}^0 = \frac{1}{-1+ik} = \frac{-i}{k+i} \quad \text{for } \operatorname{Im} k < -1$$

$\bar{f}_+(k)$ may be extended to $\mathbb{C}\setminus\{i\}$ and $\bar{f}_-(k)$ to $\mathbb{C}\setminus\{-i\}$.



(b) For $\underline{x<0}$, close contour in Uhp, with integrals around large circular arcs tending to zero as radius $R\to\infty$ (by Jordan's lemma), so

$$\frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} \bar{f}_+(k)e^{-ikx}\,dk = 0 \quad \text{(Cauchy's theorem)}$$

$$\frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} \bar{f}_-(k)e^{-ikx}\,dk = i\,.\operatorname{res}(\bar{f}_-(k)e^{-ikx}, k=-i) = i.(-ie^{-x}) = e^{-x}, \text{ as expected.} \quad \text{(C.R.T.)}$$

Similarly, for $\underline{x>0}$, close contour in Lhp, so

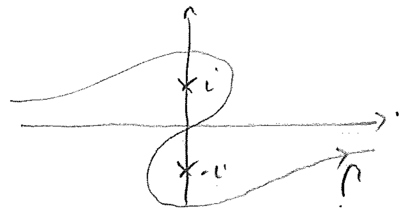
$$\frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} \bar{f}_+(k)e^{-ikx}\,dk = -i\operatorname{res}(\bar{f}_+(k)e^{-ikx}, k=i) = e^{x}, \quad \& \quad \frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} \bar{f}_-(k)e^{-ikx}\,dk = 0$$

↑ since contour is clockwise. ↑ holomorphic.

(c) $\bar{f}(k) = \bar{f}_+(k) + \bar{f}_-(k)$ for $k \in \mathbb{C}\setminus\{i,-i\}$

$$= \frac{i}{k-i} - \frac{i}{k+i}$$

$$= -\frac{2}{k^2+1}$$



An inversion contour must pass above $k=i$ (the pole from $\bar{f}_+(k)$) and below $k=-i$ (from $\bar{f}_-(k)$), as in the diagram.

Closing this contour in the Uhp for $x<0$ and the Lhp for $x>0$, we get a contribution from the residue at $k=-i$ for $x<0$ and from $k=i$ for $x>0$, exactly as in (b) for the half-range inversions.

So $\quad f(x) = \frac{1}{2\pi}\int_\Gamma \bar{f}(k)e^{-ikx}\,dk = \begin{cases} e^{-x} & x<0 \\ e^{x} & x>0 \end{cases} = e^{|x|}$

2.(a) $w(z) = \int_\Gamma g(s) e^{zs}\,ds$ in $\frac{d^2w}{dz^2} + zw = 0$ gives

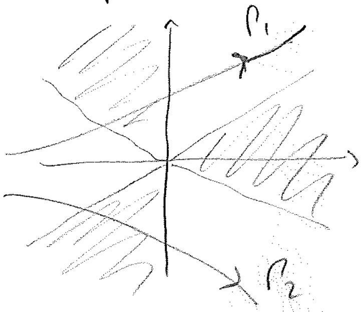
$$\int_\Gamma g(s)s^2 e^{zs} + z g(s) e^{zs}\,ds = 0$$

) integrate by parts.

$$\Rightarrow \int_\Gamma (g(s)s^2 - g'(s)) e^{zs}\,ds + [g(s)e^{zs}]_\Gamma = 0$$

This will be true for all z only if $g'(s) = s^2 g(s)$ $(\Rightarrow g = Ce^{s^3/3})$ and $[g(s)e^{zs}]_\Gamma$.

So $\boxed{w(z) = A\int_\Gamma e^{s^3/3+zs}\,ds}$ and since integrand is holomorphic we only get a non-zero w if Γ is not closed. So $[e^{s^3/3+zs}]_\Gamma = 0$ for all z, requires $e^{s^3/3+zs} \to 0$ at end points of Γ which must be at infinity, where $\mathrm{Re}(s^3/3) < 0$, ie in one of the unshaded regions on the diagram.



Γ_1 and Γ_2 are two choices for Γ that yield independent solutions of the differential equation.

(b) $w(z) = \int_\Gamma g(s) e^{zs}\,ds$ in $\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + w = 0$ gives

$$\int_\Gamma [g(s)s^2 z + g(s)s + g(s)z]\, e^{zs}\,ds = 0$$ (multiplying by z)

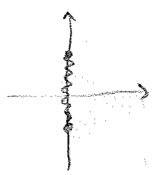
$$\Rightarrow [g(s)(s^2+1)e^{zs}]_\Gamma + \int_\Gamma [g(s)s - (g(s)(s^2+1))']\, e^{zs}\,ds = 0$$ (integrate by parts)

so we need $(g(s)(s^2+1))' = sg \Rightarrow g'(s)(s^2+1) = -sg$

$\Rightarrow \log g(s) = -\frac{1}{2}\log(s^2+1) + \text{const.}$

$\Rightarrow \boxed{g(s) = A(s^2+1)^{-1/2}}$.

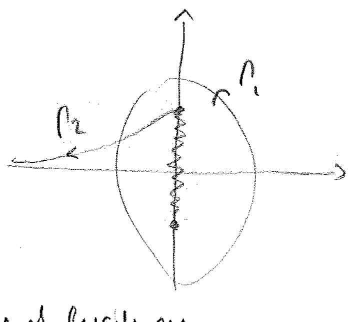
Choose branch cut to lie along $i[-1,1]$.



and $[g(s)(s^2+1)e^{zs}]_\Gamma = 0 \Rightarrow [(s^2+1)^{1/2}e^{zs}]_\Gamma = 0$.

One possibility is to take Γ closed but surrounding the branch cut. (so integrand is not holomorphic inside Γ).

Another is to go from one of the branch pts, $s = i$ say, to a point at infinity where the exponential term goes to zero. For $\mathrm{Re}(z) > 0$, going to $s = -\infty$ on the real axis achieves this.



These contours cannot be deformed onto each other and provide two independent solns of Bessels eqn

$$\boxed{w(z) = \int_\Gamma \frac{e^{zs}}{(s^2+1)^{1/2}}\,ds.}$$

[Comparing with standard integral expressions $J_0(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta)\,d\theta$, and $Y_0(z) = \frac{1}{\pi}\int_0^\infty \sin(z\sin\theta)\,d\theta - \frac{2}{\pi}\int_0^\infty e^{-z\sinh t}\,dt$, we can show that $\int_{\Gamma_1}$ gives $2\pi i J_0(z)$ and $\int_{\Gamma_2}$ gives $\frac{\pi}{2}Y_0(z) + \frac{\pi i}{2}J_0(z)$]

3. $\nabla^2 u = \frac{\partial u}{\partial x}$ $\quad y>0 \quad$ $u \to 0$ as $x^2+y^2 \to \infty$.

$u = e^{-ax}$ on $y=0$ $\quad x>0$

$\frac{\partial u}{\partial y} = 0$ $\quad y=0 \quad x<0$

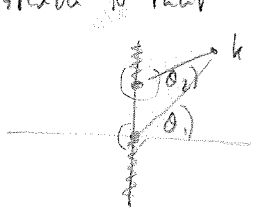
Let $f_-(x) = \begin{cases} u(x,0) & x<0 \\ 0 & x>0 \end{cases}$, $\quad g_+(x) = \begin{cases} 0 & x<0 \\ \frac{\partial u}{\partial y}(x,0) & x>0 \end{cases}$

so $u = f_-(x) + e^{-ax}H(x)$ and $\frac{\partial u}{\partial y} = g_+(x)$ on $y=0$

• Taking Fourier transform in x gives $\frac{\partial^2 \bar u}{\partial y^2} - k^2 \bar u = -ik\bar u \Rightarrow \boxed{\frac{\partial^2 \bar u}{\partial y^2} = (k^2-ik)\bar u}$ $\quad y>0$.

and the condition $\bar u \to 0$ as $y \to \infty$ means $\boxed{\bar u = A(k) e^{-(k^2-ik)^{1/2} y}}$, choosing the branch so that $(k^2-ik)^{1/2} \sim |k|$ for $k \to \pm\infty$ on real axis. i.e. $(k^2-ik)^{1/2} = |k-i|^{1/2}|k|^{1/2} e^{i(\theta_1+\theta_2)/2}$

$\theta_1 \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$, $\theta_2 \in (-\frac{3\pi}{2}, \frac{\pi}{2}]$



• Transform of conditions on $y=0 \Rightarrow \bar u(k,0) = \underbrace{\bar f_-(k)}_{\text{for Im}\,k<\beta \text{ for some } \beta} + \underbrace{\int_0^\infty e^{-ax+ikx}\,dx}_{-\frac{e^{-(a-ik)x}}{a-ik}\Big|_0^\infty = \frac{i}{k+ia} \text{ for Im}\,k>-a.}$ and $\frac{\partial \bar u}{\partial y}(k,0) = \underbrace{\bar g_+(k)}_{\text{for Im}\,k>\alpha \text{ for some } \alpha}$

so $A(k) = \bar f_-(k) + \frac{i}{k+ia}$ and $-A(k)(k^2-ik)^{1/2} = \bar g_+(k)$, for $\alpha < \text{Im}\,k < \beta$ (assuming $\alpha > -a$).

Eliminating $A(k) \Rightarrow \boxed{\frac{\bar g_+(k)}{(k^2-ik)^{1/2}} + \bar f_-(k) = -\frac{i}{k+ia}}$ for $\alpha < \text{Im}\,k < \beta$. (at this point we are assuming $\alpha<\beta$, and will later check this)

• We write $(k^2-ik)^{1/2} = \underbrace{(k-i)^{1/2}}_{'-'}\underbrace{k^{1/2}}_{'+'}$, and then $\frac{i(k-i)^{1/2}}{k+ia} = \underbrace{\frac{i(k-i)^{1/2}}{k+ia} - \frac{i(-ia-i)^{1/2}}{k+ia}}_{'-'} + \underbrace{\frac{i(-ia-i)^{1/2}}{k+ia}}_{'+'}$ $\quad (-ia-i)^{1/2} = (a+1)e^{-i\pi/4}$

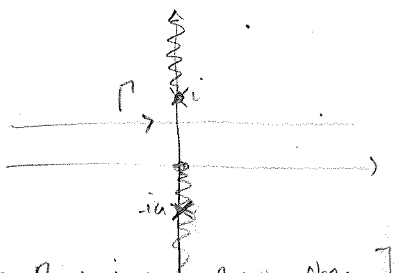
so $\underbrace{\frac{\bar g_+(k)}{k^{1/2}} + \frac{i(-ia-i)^{1/2}}{k+ia}}_{\text{holomorphic in Im}\,k>\alpha} = \underbrace{-(k-i)^{1/2}\bar f_-(k) - \frac{i(k-i)^{1/2}}{k+ia} + \frac{i(-ia-i)^{1/2}}{k+ia}}_{\text{holomorphic in Im}\,k<\beta} = E(k)$

The right hand side must be the analytic continuation of left hand side into $\text{Im}\,k<\alpha$ (since RHS and LHS are equal on overlapping strip $\alpha < \text{Im}\,k < \beta$), so together they define an entire function $E(k)$.

Assuming $\bar g_+(k) = O(\frac{1}{k^{1/2}})$ and $\bar f_-(k) = O(\frac{1}{k})$ as $k\to\infty$, we have $E(k) \to 0$ as $k \to \infty$, so Liouville $\Rightarrow$ $E(k)=0$

• Hence $\bar g_+(k) = -\frac{i(-ia-i)^{1/2}k^{1/2}}{k+ia}$, $\quad A(k) = \frac{i(-ia-i)^{1/2}}{(k+ia)(k-i)^{1/2}}$, and hence $\boxed{\bar u(k,y) = \frac{i(-ia-i)^{1/2}}{(k+ia)(k-i)^{1/2}} e^{-(k^2-ik)^{1/2}y}}$

The inversion formula gives $\boxed{u(x,y) = \frac{1}{2\pi}\int_\Gamma \frac{i(-ia-i)^{1/2}}{(k+ia)(k-i)^{1/2}} e^{-(k^2-ik)^{1/2}y - ikx}\,dk}$.



$(k-i)^{1/2} = |k-i|^{1/2} e^{i\theta_2/2}$ with $\theta_2 \in (-\frac{3\pi}{2}, \frac{\pi}{2}]$ as above.

$(-ia-i)^{1/2} = (a+1)e^{-i\pi/4}$.

(ie. $\beta=1$)

[• From this solution we can find that $f_-(x) = O(e^{x})$ and $g_+(x) = O(e^{\alpha x})$ for any $\alpha>0$, so there is an overlapping strip $\alpha < \text{Im}\,k < \beta$]

4. $\nabla^2 u = u \quad y>0 \qquad u \to 0$ as $y \to \infty$

$u = x$ on $y=0,\ x>0$

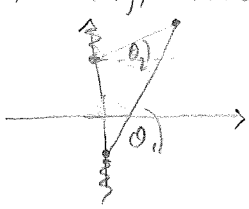
$\frac{\partial u}{\partial y} = 0$ on $y=0 \quad x<0$

Define $f_-(x) = \begin{cases} u(x,0) & x<0 \\ 0 & x>0 \end{cases}$ and $g_+(x) = \begin{cases} 0 & x<0 \\ \frac{\partial u}{\partial y}(x,0) & x>0 \end{cases}$, 10.

$u(x,0) = xH(x) + f_-(x)$ and $\frac{\partial u}{\partial y}(x,0) = g_+(x)$

Taking Fourier transforms in x, $\frac{d^2\bar{u}}{dy^2} - k^2\bar{u} = \bar{u}$. As $\bar{u} \to 0$ as $y \to \infty$ $\Rightarrow$ $\bar{u}(k,y) = A(k)e^{-(k^2+1)^{1/2}y}$

where $(k^2+1)^{1/2} = |k+i|^{1/2}|k-i|^{1/2}e^{i(\theta_1+\theta_2)/2}$, and $\theta_1 \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$, $\theta_2 \in (-\frac{3\pi}{2}, \frac{\pi}{2}]$



for Im $k<\beta$

Then $\bar{u}(k,0) = \underbrace{\int_0^\infty x e^{ikx}\,dx}_{} + \bar{f}_-(k)$ and $\frac{\partial \bar{u}}{\partial y}(k,0) = \bar{g}_+(k)$

for Im $k > \alpha$.

$\frac{1}{ik}xe^{ikx}\Big|_0^\infty + \frac{1}{k^2}e^{ikx}\Big|_0^\infty$

$= -\frac{1}{k^2}$ for Im $k>0$.

so $A(k) = -\frac{1}{k^2} + \bar{f}_-(k)$ and $-A(k)(k^2+1)^{1/2} = \bar{g}_+(k)$

$\Rightarrow \quad (k^2+1)^{1/2}\bar{f}_-(k) + \bar{g}_+(k) = \frac{(k^2+1)^{1/2}}{k^2}$ on some hopefully overlapping strip $\max(0,\alpha) < \mathrm{Im}\,k < \beta$

Taking $(k+i)^{1/2}$ and $(k-i)^{1/2}$ defined in the obvious way given the branch cuts above, we have

$(-i)^{1/2}(1+ik)^{1/2}$

$$\frac{\bar{g}_+(k)}{(k+i)^{1/2}} + (k-i)^{1/2}\bar{f}_-(k) = \frac{(k-i)^{1/2}}{k^2} = \underbrace{\frac{(k-i)^{1/2}}{k^2} - \frac{(-i)^{1/2}(1+\frac{1}{2}ik)}{k^2}}_{-G_-(k)} + \underbrace{\frac{(-i)^{1/2}(1+\frac{1}{2}ik)}{k^2}}_{G_+(k)}$$

ie 'removing' the singularity. $[(-i)^{1/2} = e^{-i\pi/4}]$

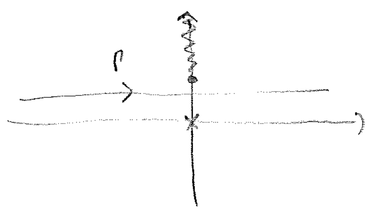
ie. $$\boxed{\frac{\bar{g}_+(k)}{(k+i)^{1/2}} - \frac{(-i)^{1/2}(1+\frac{1}{2}ik)}{k^2} = -(k-i)^{1/2}\bar{f}_-(k) + \frac{(k-i)^{1/2}}{k^2} - \frac{(-i)^{1/2}(1+\frac{1}{2}ik)}{k^2}} = E(k).$$

- LHS is holomorphic in Im $k>\alpha$, say, and RHS in Im $k<\beta$, so assuming some overlap, they are analytic continuation of each other onto the whole of $\mathbb{C}$, so define an entire function $E(k)$.

Assuming $\bar{g}_+(k) = O(\frac{1}{k^{1/2}})$ and $\bar{f}_-(k) = O(\frac{1}{k^{3/2}})$, $E(k) \to 0$ as $k \to \infty$, so $E(k)=0$ by Liouville.

Hence. $\bar{g}_+(k) = \frac{(-i)^{1/2}(1+\frac{1}{2}ik)(k+i)^{1/2}}{k^2}$, $\quad A(k) = -\frac{(-i)^{1/2}(1+\frac{1}{2}ik)}{k^2(k-i)^{1/2}}$, and hence

$$\boxed{u(x,0) = -\frac{(-i)^{1/2}}{2\pi}\int_\Gamma \frac{1+\frac{1}{2}ik}{k^2(k-i)^{1/2}}e^{-ikx}\,dk.}$$

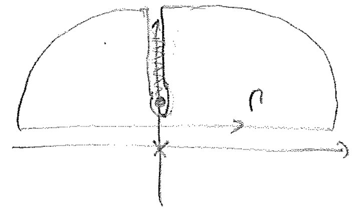


• For $x > 0$, close in lhp, accounting for residue at $k=0$, and noting integral around large semicircle tends to zero, so
$$u(x,0) = -\frac{(-i)^{1/2}}{2\pi} \cdot \underset{\substack{\uparrow \\ \text{since clockwise} \\ \text{contour.}}}{-2\pi i} \cdot \operatorname{res}\left(\frac{1+\frac{1}{2}ik}{k^2(k-i)^{1/2}} e^{-ikx}, k=0 \right) = (-i)^{1/2} i \left(\frac{-ix}{(-i)^{1/2}} \right) = x. \checkmark$$

$$\left[\text{residue} = \frac{d}{dk}\left(\frac{(1+\frac{1}{2}ik)e^{-ikx}}{(k-i)^{1/2}} \right)\Bigg|_{k=0} = \frac{\frac{1}{2}ie^{-ikx} - ix(1+\frac{1}{2}ik)e^{-ikx}}{(k-i)^{1/2}} - \frac{(1+\frac{1}{2}ik)e^{-ikx}}{2(k-i)^{3/2}} \Bigg|_{k=0} = \frac{\frac{1}{2}i - ix}{(-i)^{1/2}} - \frac{i}{2(-i)^{1/2}} = \frac{-ix}{(-i)^{1/2}} \right]$$

$$\left[\text{or } \frac{(1+\frac{1}{2}ik)(1+ik)^{-1/2}(1-ikx+\dots)}{(-i)^{1/2}} = \frac{1 + \cancel{\frac{1}{2}ik} - \cancel{\frac{1}{2}ik} - ikx + O(k^2)}{(-i)^{1/2}} \Rightarrow \text{residue } -\frac{ix}{(-i)^{1/2}}. \right]$$

• For $x < 0$, close in Uhp, diverting around the branch cut, and noting that integrals around circular arcs tend to zero, so



$$u(x,0) = -\frac{(-i)^{1/2}}{2\pi} \cdot \underbrace{\left(\int_{\substack{\text{RHS of} \\ \text{cut}}} - \int_{\substack{\text{LHS of} \\ \text{cut}}} \right)}_{} \frac{1+\frac{1}{2}ik}{k^2(k-i)^{1/2}} e^{-ikx} \, dk.$$

$$= 2\int_{\substack{\text{RHS of} \\ \text{cut}}} \quad \text{on which } k = it, \; dk = i\,dt, \; (k-i)^{1/2} = e^{i\pi/4}(t-1)^{1/2}.$$

$$= + \frac{\cancel{e^{-i\pi/4}}}{2\pi} \, 2 \int_1^\infty \frac{1-\frac{1}{2}t}{(-t^2)(t-1)^{1/2}\cancel{e^{i\pi/4}}} e^{tx} \, (i\,dt)$$

$$= \frac{1}{\pi} \int_1^\infty \frac{1-\frac{1}{2}t}{t^2(t-1)^{1/2}} e^{tx} \, dt.$$

[Note that $f_-(x) = O(e^{x})$ (so $\beta = 1$) and $g_+(x) = O(e^{\alpha x})$ for any $\alpha > 0$, so the assumed overlapping strip $\alpha < \operatorname{Im} k < \beta$ on which the relevant $+/-$ functions were both holomorphic does indeed exist]