Sheet 0: Revision of core complex analysis

These questions may be attempted before the start of term, but the first few lectures will cover this material so you may prefer to do them during weeks 1 and 2, alongside Sheet 1. Solutions are available online.

Q1 (a) By treating $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$ as independent variables and using the chain rule write

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} \qquad \text{in terms of} \qquad \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \overline{z}}.$$

(b) Show that the Cauchy–Riemann equations for a holomorphic function $f(z, \overline{z}) = u(x, y) + \mathrm{i}v(x, y)$ (with u, v real) are equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

and that we then have

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{1}{\mathrm{i}}\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}.$$

(c) Calculate $\partial \overline{f(z)}/\partial z$ and deduce that $\overline{f}(z) = \overline{f(\overline{z})}$ is holomorphic. Show also that

$$\frac{\partial \overline{f(z)}}{\partial \overline{z}} = \overline{f'(z)}.$$

(d) Finally, show that Laplace's equation $u_{xx} + u_{yy} = 0$ is equivalent to $4u_{z\overline{z}} = 0$, and deduce that any real solution of Laplace's equation may be written in the form

$$u(x, y) = f(z) + \overline{f(z)}$$

for some holomorphic function $f(z)$.

Q2 (a) Show that, if $\zeta^2 = z^2 + 1$ and $z = \mathrm{i} + r_1 \mathrm{e}^{\mathrm{i}\theta_1} = -\mathrm{i} + r_2 \mathrm{e}^{\mathrm{i}\theta_2}$, where $r_1,\ r_2 > 0$, $\theta_1,\ \theta_2 \in \mathbb{R}$, then

$$\zeta = \pm (r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)/2}.$$

Explain why $z = \pm\mathrm{i}$ are the branch points of the multifunction $f(z) = (z^2 + 1)^{1/2}$.

(b) Consider the branch of $f(z) = (r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}$, with $-\pi/2 < \theta_1,\ \theta_2 \leq 3\pi/2$. State the values of $(\theta_1 + \theta_2)/2$ on either side of the imaginary axis, and hence compute $f(\pm 0 + \mathrm{i}y)$ in terms of y for $y \in \mathbb{R}$. Deduce that the branch cut is $S = \{x + \mathrm{i}y : x = 0,\ |y| \leq 1\}$. Show that $f(z) \sim z$ as $|z| \to \infty$ (*i.e.* $f(z)/z \to 1$ as $|z| \to \infty$). Sketch the image of the cut z-plane $\mathbb{C}\backslash S$ under the map $\zeta = f(z)$.

(c) Consider the branch of $f(z) = +(r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}$, with $-3\pi/2 < \theta_1 \leq \pi/2$ and $-\pi/2 < \theta_2 \leq 3\pi/2$. State the values of $(\theta_1 + \theta_2)/2$ on either side of the imaginary axis, and hence compute $f(\pm 0 + \mathrm{i}y)$ in terms of y for $y \in \mathbb{R}$. Deduce that the branch cut is along the imaginary axis from $z = -\mathrm{i}\infty$ to $z = -\mathrm{i}$ and from $z = \mathrm{i}$ to $z = \mathrm{i}\infty$. Compute $f(x)$ in terms of x for $x \in \mathbb{R}$. Show that

$$f(z) \sim \begin{cases} z & \text{as } |z| \to \infty \text{ with } \mathrm{Re}(z) > 0, \\ -z & \text{as } |z| \to \infty \text{ with } \mathrm{Re}(z) < 0. \end{cases}$$

Sketch the images of the half-planes $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(z) < 0$ under the map $\zeta = f(z)$.

Q3 Use contour integration to evaluate

$$\int_{-1}^{1} \frac{\sqrt{1 - x^2}}{x^2 + 1}\, \mathrm{d}x.$$

[*Hint: Consider the integral of* $\left(z^2 - 1\right)^{1/2} / \left(z^2 + 1\right)$ *with branch cut from* $z = -1$ *to* $z = 1$*, around a large circle.*]

Q4 Find the Fourier transforms of (a) $\mathrm{e}^{-|x|}$ and (b) e^{-x^2} (for the latter you may assume that $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$). Verify the inverses by contour integration.

[*Hint: (b) Integrate round a rectangular contour.*]