1. (a) $z = x+iy$, $\bar z = x - iy$ so $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar z}$ $\quad \dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar z}\right)$

Conversely $x = \frac{1}{2}(z+\bar z)$, $y = \frac{1}{2}(z - \bar z)$ so $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$ $\quad \dfrac{\partial}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

(b) If $f(z,\bar z) = u(x,y) + iv(x,y)$ then $\dfrac{\partial f}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)(u+iv) = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right) + \dfrac{i}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right)$

so $\dfrac{\partial f}{\partial \bar z} = 0 \iff \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ i.e. the Cauchy Riemann equations.

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

by C.R.E.

(c) $\dfrac{\partial}{\partial z}\overline{f(z)} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)(u-iv) = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right) - \dfrac{i}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) = 0.$

Replacing z with $\bar z$, $\dfrac{\partial}{\partial \bar z}\overline{f(\bar z)} = 0$ so $\overline{f(\bar z)} = \bar f(z)$ is holomorphic by (b).

$$\frac{\partial}{\partial \bar z}\overline{f(z)} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u-iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} = \overline{f'(z)}$$

(d) $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = \dfrac{\partial^2}{\partial z^2} + 2\dfrac{\partial^2}{\partial z \partial \bar z} + \dfrac{\partial^2}{\partial \bar z^2} - \left(\dfrac{\partial^2}{\partial z^2} - \dfrac{2\partial^2}{\partial z \partial \bar z} + \dfrac{\partial^2}{\partial \bar z^2}\right) = 4\dfrac{\partial^2}{\partial z \partial \bar z}$, so $u_{xx} + u_{yy} = 4u_{z\bar z}$

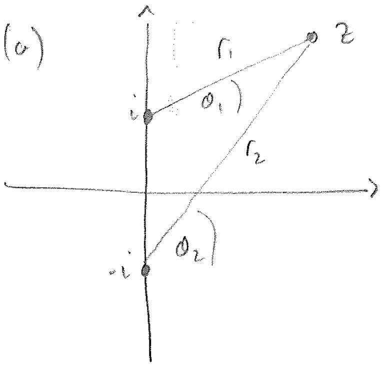
$u_{z\bar z} = 0 \Rightarrow u_z = g'(z)$ for some $g'(z)$ holomorphic $\Rightarrow u = g(z) + h(\bar z)$ for some holomorphic g, h.

But if u is real, we can write $u = \frac{1}{2}(u + \bar u) = \frac{1}{2}\left(g(z) + h(\bar z) + \overline{g(z)} + \overline{h(\bar z)}\right)$

$$= \frac{1}{2}\Big(\underbrace{g(z) + \overline{h(\bar z)}}_{2f(z)} + \underbrace{\overline{g(z) + \overline{h(\bar z)}}}_{2\overline{f(z)}}\Big) = f(z) + \overline{f(z)}$$

i.e. this combination is holomorphic, so call it $2f(z)$.

2. (a)



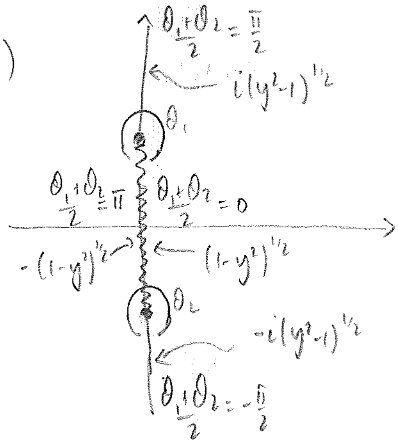
$\zeta^2 = z^2+1 = (z+i)(z-i) = r_1 r_2 e^{i(\theta_1+\theta_2)}$

so $\zeta = \pm (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}$ (± corresponds to choice of θ_1 & θ_2; see below)

$\zeta = \pm i$ are branch points since if we make a circuit around a circular contour $\gamma(\pm i, \varepsilon)$ the value of ζ changes (since θ_1 or θ_2 increases by 2π, so ζ changes by factor of -1).

No other points are branch points since if we make a similar circuit around $\gamma(a,\varepsilon)$ for $a \neq \pm i$ we get the same value for ζ (provided ε is taken sufficiently small).

(b)

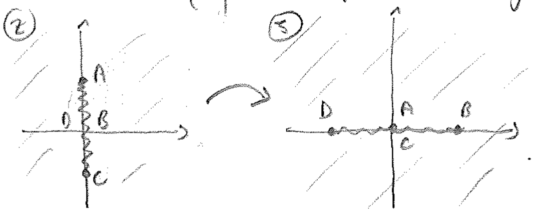


Take $\boxed{\theta_1 \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \quad \theta_2 \in (-\frac{\pi}{2}, \frac{3\pi}{2}], \ \zeta = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}}$

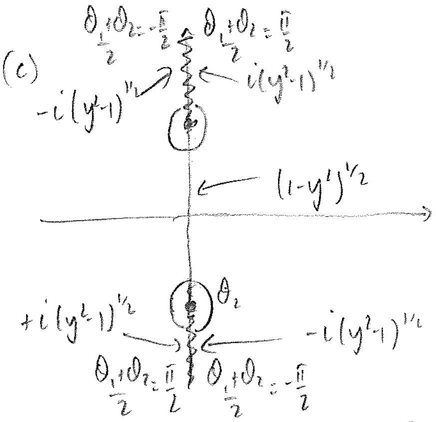
For large $z = re^{i\theta}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, we have

$\theta_1, \theta_2 \sim \theta \quad r_1, r_2 \sim r$, so $\zeta \sim re^{i\theta} = z$.

(regardless of 'where' we go to infinity).



(c)

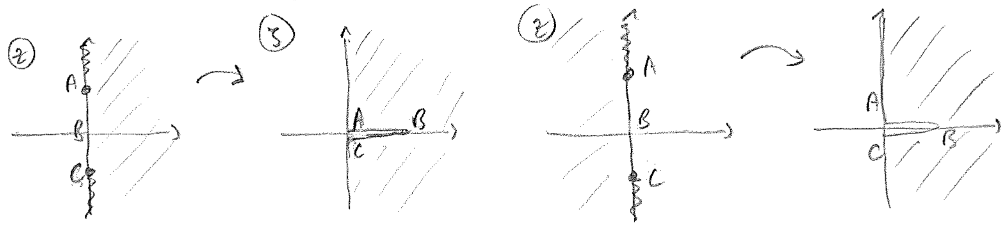


Take $\boxed{\theta_1 \in (-\frac{3\pi}{2}, \frac{\pi}{2}] \quad \theta_1 \in (-\frac{\pi}{2}, \frac{3\pi}{2}] \quad \zeta = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}}$

For large $z = re^{i\theta}$ with $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$, we have

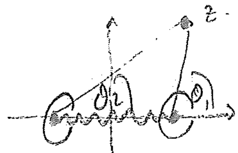
$\theta_1 \sim \begin{cases} \theta & \text{if in RHP} \\ \theta - 2\pi & \text{if in LHP} \end{cases} \quad \theta_2 \sim \theta \quad r_1, r_2 \sim r$

so $z \sim \begin{cases} re^{i\theta} = z & \text{in RHP} \\ re^{i\theta - i\pi} = -z & \text{in LHP} \end{cases}$

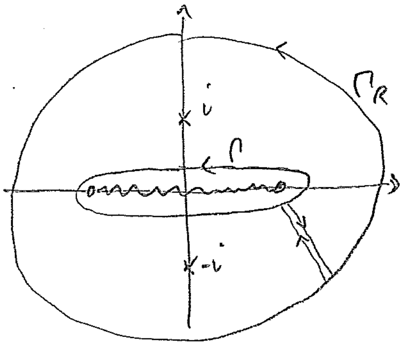


3. Consider $f(z) = \dfrac{(z^2-1)^{1/2}}{z^2+1}$ with $(z^2-1)^{1/2} = |z^2-1|^{1/2} e^{i(\theta_1+\theta_2)/2}$

$\theta_1 = \arg(z-1) \in [0, 2\pi)$
$\theta_2 = \arg(z+1) \in [0, 2\pi)$



(range is not important provided they're the same, so cut is along $[-1,1]$.)



Consider $\int_{\Gamma_R} - \int_\Gamma f(z)\,dz = 2\pi i\left[\operatorname{res}(f,i) + \operatorname{res}(f,-i)\right]$ by C.R.T.

Note $\int_\Gamma f(z)\,dz = \underset{\substack{\text{top}\\ \theta_1=\pi,\theta_2=0}}{\int} + \underset{\substack{\text{bottom}\\ \theta_1=\pi,\theta_2=2\pi}}{\int} = \int_1^{-1} \dfrac{i\sqrt{1-x^2}}{x^2+1}\,dx + \int_{-1}^{1} \dfrac{-i\sqrt{1-x^2}}{x^2+1}\,dx = \boxed{-2i\int_{-1}^{1} \dfrac{\sqrt{1-x^2}}{x^2+1}\,dx}$

Also, $\underset{\frac{\theta_1+\theta_2}{2} = \frac{\pi}{2}}{\operatorname{res}(f,i)} = \dfrac{i\sqrt{2}}{2i} = \boxed{\dfrac{1}{\sqrt{2}}}$ & $\underset{\frac{\theta_1+\theta_2}{2} = \frac{3\pi}{2}}{\operatorname{res}(f,-i)} = \dfrac{-i\sqrt{2}}{-2i} = \boxed{\dfrac{1}{\sqrt{2}}}$

since simple pole, $2z\big|_{z=i}$

For large z, $(z^2+1)^{1/2} \sim z$, so $f(z) \sim \frac{1}{z}$. Hence $\boxed{\int_{\Gamma_R} f(z)\,dz \sim \int_{\Gamma_R} \frac{1}{z}\,dz = 2\pi i}$ as $R \to \infty$.

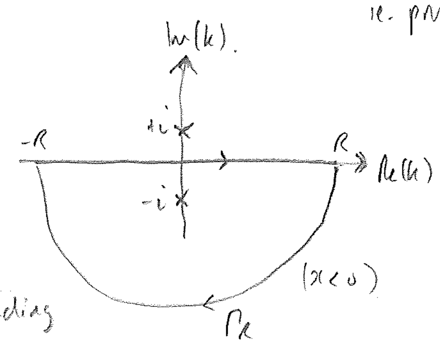
So putting together we have $2\pi i + 2i\int_{-1}^{1} \dfrac{\sqrt{1-x^2}}{x^2+1}\,dx = 2\pi i\left(\dfrac{2}{\sqrt{2}}\right) \Rightarrow \boxed{\int_{-1}^{1} \dfrac{\sqrt{1-x^2}}{x^2+1}\,dx = \pi(\sqrt{2}-1)}$

4. (a) $\mathcal{F}(e^{-|x|}) = \int_{-\infty}^{\infty} e^{-|x|} e^{ikx}\, dx = \int_{0}^{\infty} e^{-x+ikx} + e^{-x-ikx}\, dx = \frac{1}{1-ik} + \frac{1}{1+ik} = \frac{2}{1+k^2}$

provided $Re(1-ik)>0$ and $Re(1+ik)>0$
i.e. provided $|Im(k)|<1$.

For the inverse,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+k^2} e^{-ikx}\, dk.$$



close the contour with a semi-circle in lower half plane or upper half plane depending on sign of x.

For $x>0$, close in lower half plane, so integral around Γ_R has $\left|\int_{\Gamma_R} \frac{2}{1+k^2} e^{-ikx}\, dk\right| \leq \pi R \cdot \frac{2}{R^2-1} \to 0$ as $R\to\infty$ (estimation theorem)

By Cauchy's residue theorem, $\int_{\Gamma} \frac{2}{1+k^2} e^{-ikx}\, dk = 2\pi i \cdot (\text{residue at } k=-i) = 2\pi i \cdot \frac{2}{-2i} e^{-x} = -2\pi e^{-x}$

(where Γ is the whole closed contour), and hence, noting the direction of integration, $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+k^2} e^{-ikx}\, dk = e^{-x}$.

Similarly for $x<0$ (or write $x=-\hat{x}$ and take complex conjugate of above result), we find $f(x)=e^{x}$.

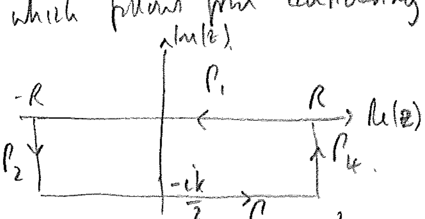
Hence $f(x) = e^{-|x|}$, as expected.

(b) $\mathcal{F}(e^{-x^2}) = \int_{-\infty}^{\infty} e^{-x^2+ikx}\, dx = \int_{-\infty}^{\infty} e^{-(x-\frac{1}{2}ik)^2 - \frac{1}{4}k^2}\, dx \overset{(*)}{=} \sqrt{\pi}\, e^{-\frac{1}{4}k^2}$

Similarly, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{\pi}\, e^{-\frac{1}{4}k^2 - ikx}\, dk \overset{k=2\tilde{k}}{=} \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\tilde{k}+ix)^2 - x^2}\, 2d\tilde{k} \overset{(*)}{=} e^{-x^2}$, as expected

(*) Here, we have used the result that $\int_{-\infty}^{\infty} e^{-(x-\frac{1}{2}ik)^2}\, dx = \sqrt{\pi}$, which follows from considering

$\int_{\Gamma} e^{-z^2}\, dz = 0$ (by Cauchy's theorem) for Γ the rectangular contour:



A standard result gives $\int_{\Gamma_1} = -\sqrt{\pi}$. $\int_{\Gamma_2}$ and $\int_{\Gamma_4}$ both tend to zero as $R\to\infty$ (since $|e^{-z^2}|\to 0$ when $Re(z)\to\pm\infty$). Hence $\int_{\Gamma_3} = -\int_{\Gamma_1} = \sqrt{\pi}$, and $\int_{\Gamma_3} \underset{z=x-\frac{ik}{2}}{=} \int_{-\infty}^{\infty} e^{-(x-\frac{ik}{2})^2}\, dx$ is the desired integral.