Sheet 1: Conformal mapping, Schwarz–Christoffel, boundary value problems

Q1 (a) Find the image of the common part of the discs $|z-1|<1$ and $|z+\mathrm{i}|<1$ under the mapping $\zeta = 1/z$.

(b) Find the image of the strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$ under the map $\zeta = \sin z$.

(c) Find the image of the strip $-\pi < y < \pi$ under the map $\zeta = z + \mathrm{e}^z$.

[*Hint: Where are the critical points of the map?*]

Q2 (a) D is the region exterior to the two circles $|z-1| = 1$ and $|z+1| = 1$. Find a conformal mapping from D to the exterior of the unit circle.

[*Hint: First apply an inversion with respect to the origin ($z \mapsto 1/z$), then rotate and scale so as to use the exponential function to get to a half plane, from where use Möbius.*]

(b) Find a conformal map from the unit disc $|z| < 1$ onto the strip $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$, taking the origin to the origin, 1 to $\xi = +\infty$, -1 to $\xi = -\infty$. (Thus, the upper half of the unit circle maps to $\eta = \frac{\pi}{2}$ and the lower half to $\eta = -\frac{\pi}{2}$.)

(c) Find a conformal map of the quarter-disc $0 < |z| < 1$, $0 < \arg z < \frac{\pi}{2}$ to the upper half-plane $\eta > 0$, taking 0 to 0, 1 to 1, and i to ∞.

Q3 Write down, as an integral, the Schwarz–Christoffel map from a half-plane to a rectangle, with the vertices being the images of the points $z = \pm 1$ and $z = \pm a$, where $a > 1$ is real. Explain why a cannot be specified arbitrarily, but is determined by the aspect ratio of the rectangle.

Q4 The domain D consists of the upper half-plane with a solid wall along the real axis. The segment of the imaginary axis from $z = 0$ to $z = \mathrm{i}$ is also impermeable to fluid. Find the complex potential for inviscid incompressible irrotational flow in D with velocity $(U_1, 0)$ at infinity.

Q5 The domain D consists of the right-hand half plane $x > 0$ with the circle $|z-a| = b$, $0 < b < a$, and its interior removed. Find the temperature $u(x, y)$ in steady heat flow if $u = 0$ on the y axis, $u = 1$ on $|z-a| = b$, and $u \to 0$ at infinity.

[*Hint: Show that the mapping $\zeta = (z-\alpha)/(z+\alpha)$, with α real and positive, takes D onto an annular region with the imaginary axis mapping to $|\zeta| = 1$ and show that, if $\alpha^2 = a^2 - b^2$, then the image of D is a concentric circular annulus.*]

Q6 (a) Carefully define a branch of the function $\cosh^{-1}(Z)$ that is holomorphic in the upper half-plane. What is $\cosh^{-1}(0)$? What is the derivative of $\cosh^{-1}(Z)$?

(b) Show that the Schwarz–Christoffel map from the upper half-plane to the exterior of the half-strip $0 < x < \infty$, $-1 < y < 1$ has the form

$$z = A + C\left(Z\sqrt{Z^2-1} - \cosh^{-1} Z\right),$$

and find the constants A and C.

[*Hint: Map $Z = \pm 1$ to the finite corners of the domain, and $Z = \infty$ to the vertex at $x = \infty$.*]

(c) Hence find the complex potential $w(z)$ for potential flow past this obstacle with a uniform stream $(U_1, 0)$ at infinity.

[*Hint: Bearing in mind the behaviour of the mapping at infinity, think carefully about the potential in the Z plane: it is not 'constant $\times$ Z.'*]