Sheet 3: Plemelj formulae and applications

Q1 State the Plemelj formulae for the function defined by

$$w(z)=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta-z},$$

where Γ is a contour in the complex plane.

By defining an appropriate branch of $w(z)=(z-1)^{\alpha-1}/(z+1)^{\alpha}$, where $0<\alpha<1$, and using the Plemelj formulae, evaluate

$$⨍_{-1}^{1}\frac{(1-t)^{\alpha-1}\,\mathrm{d}t}{(1+t)^{\alpha}(t-x)}\quad\text{for}\quad -1<x<1.$$

Q2 Many mechanics problems lead to the problem of finding $\phi(x,y)$ such that $\nabla^2\phi=0$ except on $y=0$, $0\le x\le c$; $\lim_{y\downarrow\uparrow 0}\partial\phi/\partial y=g_{\pm}(x)$ for $0<x<c$, where $g_{\pm}(x)$ is continuous on $0\le x\le c$; $|\nabla\phi|$ is finite or has an inverse square-root singularity at $(0,0)$ and $(c,0)$; and $|\nabla\phi|\to 0$ as $x^2+y^2\to\infty$.

If $w(z)=-\mathrm{d}(\phi+\mathrm{i}\psi)/\mathrm{d}z$, where ψ is the harmonic conjugate of ϕ, then (I) w is holomorphic except on $\bar{\Gamma}=\{x+\mathrm{i}y:0\le x\le c,\ y=0\}$; (II) $\operatorname{Im}w_{+}=g_{+}$, $\operatorname{Im}w_{-}=g_{-}$ on the contour $\Gamma=\{x+\mathrm{i}y:0<x<c,\ y=0\}$, where $g_{\pm}$ is continuous on $\bar{\Gamma}$; (III) w is finite or has an inverse square-root singularity at $z=0$ and $z=c$; and (IV) $w\to 0$ as $z\to\infty$.

(a) Suppose $g_{+}(x)=-g_{-}(x)$. Use (I) and (II) to deduce that a possible solution is given by

$$w(z)=\frac{1}{\pi}\int_0^c\frac{g_{+}(\xi)\,\mathrm{d}\xi}{\xi-z}+h(z),$$

where $h(z)$ is an arbitrary function of z that is holomorphic on $\mathbb{C}\backslash\{0,\,c\}$ and real on Γ. Use (III), (IV) and Liouville's theorem to deduce that $h=0$.

(b) Now suppose that $g_{+}(x)=g_{-}(x)=g(x)$. Show that, if $\widetilde{w}(z)$ is holomorphic and non-zero away from $\bar{\Gamma}$, with $\widetilde{w}_{+}(x)=-\widetilde{w}_{-}(x)\neq 0$ on Γ, then a possible solution for $w(z)$ is given by

$$\frac{w(z)}{\widetilde{w}(z)}=\frac{1}{\pi}\int_0^c\frac{g(\xi)\,\mathrm{d}\xi}{\widetilde{w}_{+}(\xi)(\xi-z)}+H(z),$$

where $H(z)$ is an arbitrary function of z holomorphic on $\mathbb{C}\backslash\{0,\,c\}$.

(c) An aerofoil problem has $g_{+}(x)=g_{-}(x)=-\alpha$, constant, together with the requirement that w has an inverse square-root singularity at $z=0$, is finite at $z=c$, and $w\to 0$ as $z\to\infty$. By defining the branch of $\widetilde{w}(z)=(c-z)^{1/2}\,z^{-1/2}$ that has a cut along Γ and satisfies $\widetilde{w}_{+}(\xi)>0$ for $0<\xi<c$, show that a possible solution is given by

$$w(z)=-\frac{\alpha\,(c-z)^{1/2}}{\pi z^{1/2}}\left[\int_0^c\frac{\xi^{1/2}\,\mathrm{d}\xi}{(c-\xi)^{1/2}\,(\xi-z)}+H(z)\right],$$

and determine $H(z)$.

(d) By relating the integral term to a suitable contour integral that can be deformed to a large circle, show that

$$w(z)=-\alpha\left(\frac{c-z}{z}\right)^{1/2}-i\alpha.$$

Noting that the behaviour of w at infinity is related to the circulation $\tilde{\Gamma}$ about the aerofoil by $w\sim\mathrm{i}\tilde{\Gamma}/2\pi z$, deduce that the circulation in this case is $\tilde{\Gamma}=-\pi c\alpha$.

Q3 Suppose f satisfies the Cauchy singular integral equation

$$a(t)f(t) + \frac{b(t)}{\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t} = c(t) \qquad \text{on} \quad \Gamma, \tag{$\star$}$$

where a, b and c are holomorphic in a neighbourhood of Γ.

(a) Show that, if

$$w(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

then $(a+b)w_+ + (b-a)w_- = c$ on Γ.

(b) Now suppose $a+b$ and $a-b$ are not zero on Γ, and that $\tilde{w}$ is holomorphic and non-zero away from Γ and that $(a+b)\tilde{w}_+ = -(b-a)\tilde{w}_- \neq 0$ on Γ. Show that

$$\left(\frac{w}{\tilde{w}}\right)_+ - \left(\frac{w}{\tilde{w}}\right)_- = \frac{c}{(a+b)\tilde{w}_+} \qquad \text{on} \quad \Gamma.$$

(c) Hence show that

$$w(z) = \frac{\tilde{w}(z)}{2\pi\mathrm{i}} \int_\Gamma \frac{c(\zeta)\,\mathrm{d}\zeta}{(a(\zeta)+b(\zeta))\tilde{w}_+(\zeta)(\zeta - z)}$$

is a possible solution for $w(z)$ and that $(\star)$ is satisfied by

$$f(t) = -\frac{b(t)\tilde{w}_+(t)}{(a(t)-b(t))}\frac{1}{\pi\mathrm{i}} \fint_\Gamma \frac{c(\zeta)}{(a(\zeta)+b(\zeta))\tilde{w}_+(\zeta)}\frac{\mathrm{d}\zeta}{\zeta - t} + \frac{c(t)a(t)}{a(t)^2 - b(t)^2}.$$