Sheet 4: Transforms, Wiener-Hopf

Q1 Suppose $f(x) = e^{|x|}$ for $-\infty < x < \infty$.

(a) If $f(x) = f_+(x) + f_-(x)$, where $f_+(x) = 0$ for $x < 0$ and $f_-(x) = 0$ for $x > 0$, show that the Fourier transforms of f_+ and f_- are given by

$$\bar{f}_+(k) = \int_{-\infty}^{\infty} f_+(x)e^{ikx}dx = \frac{i}{k-i} \quad \text{for} \quad \text{Im}(k) > 1$$

and

$$\bar{f}_-(k) = \int_{-\infty}^{\infty} f_-(x)e^{ikx}dx = -\frac{i}{k+i} \quad \text{for} \quad \text{Im}(k) < -1$$

To which parts of the complex k-plane may $\bar{f}_\pm(k)$ be analytically continued?

(b) Use contour integration to evaluate

$$\frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} \bar{f}_+(k)e^{-ikx}dk \quad \text{and} \quad \frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} \bar{f}_-(k)e^{-ikx}dk$$

for $x < 0$ and $x > 0$, where $\alpha > 1$ and $\beta < -1$.

(c) Over which part of the complex k-plane is it possible to define $\bar{f}(k)$? Sketch a suitable inversion contour Γ for which

$$f(x) = \frac{1}{2\pi}\int_\Gamma \bar{f}(k)e^{-ikx}dk.$$

Verify this result using contour integration.

Q2 (a) Show that

$$w(z) = \int_\Gamma g(\zeta)e^{z\zeta}\, d\zeta \qquad (*)$$

is a solution of Airy's equation

$$\frac{d^2w}{dz^2} + zw = 0$$

only if $g(\zeta) = Ae^{\zeta^3/3}$ and $[g(\zeta)e^{z\zeta}]_\Gamma = 0$, where A is a constant. Identify two choices for Γ which lead to two independent solutions of the differential equation.

(b) Show that $(*)$ is a solution of Bessel's equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + w = 0$$

only if $g(\zeta) = A/(1+\zeta^2)^{1/2}$ and $[(1+\zeta^2)g(\zeta)e^{z\zeta}]_\Gamma = 0$. Identify two choices for Γ which lead to two independent solutions of the differential equation for $\text{Re}(z) > 0$.

Q3 Suppose $u(x, y)$ satisfies the mixed boundary value problem

$$\nabla^2 u = \frac{\partial u}{\partial x} \quad \text{in } y > 0,$$

with

$$u = e^{-ax} \quad \text{on } y = 0,\ x > 0, \qquad\qquad \frac{\partial u}{\partial y} = 0 \quad \text{on } y = 0,\ x < 0,$$

where $a > 0$, and $u \to 0$ as $x^2 + y^2 \to \infty$. Define

$$f_-(x) = \begin{cases} u(x,0) & x < 0, \\ 0 & x > 0, \end{cases} \qquad\qquad g_+(x) = \begin{cases} 0 & x < 0, \\ \dfrac{\partial u}{\partial y}(x,0) & x > 0. \end{cases}$$

Show that the Fourier transforms $\bar{f}_-(k)$ and $\bar{g}_+(k)$ of $f_-(x)$ and $g_+(x)$ respectively satisfy

$$\frac{1}{(k^2 - \mathrm{i}k)^{1/2}}\,\bar{g}_+(k) + \bar{f}_-(k) = -\frac{\mathrm{i}}{k + \mathrm{i}a},$$

where you should define precisely the branch of the multifunction $\left(k^2 - \mathrm{i}k\right)^{1/2}$. Deduce that

$$u(x,y) = \frac{1}{2\pi}\int_\Gamma \frac{\mathrm{i}(-\mathrm{i}a - \mathrm{i})^{1/2}}{(k + \mathrm{i}a)(k - \mathrm{i})^{1/2}}\,\mathrm{e}^{-y\left(k^2 - \mathrm{i}k\right)^{1/2} - \mathrm{i}kx}\,\mathrm{d}k,$$

where you should define precisely the branch of $(k - \mathrm{i})^{1/2}$, the appropriate value of $(-\mathrm{i}a - \mathrm{i})^{1/2}$ and a suitable inversion contour Γ.

[*You may assume without proof that $k^{1/2}\bar{g}_+(k)$ and $k\bar{f}_-(k)$ are bounded as $k \to \infty$.*]

Q4 Suppose $u(x,y)$ satisfies partial differential equation $\nabla^2 u = u$ in $y > 0$ subject to the mixed boundary conditions

$$\frac{\partial u}{\partial y}(x,0) = 0 \quad \text{for } x < 0, \qquad\qquad u(x,0) = x \quad \text{for } x > 0,$$

and $u(x,y) \to 0$ as $y \to \infty$. By taking a Fourier transform and using the Wiener–Hopf method, show that

$$u(x,0) = \frac{1}{2\pi}\int_\Gamma A(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \qquad \text{where} \quad A(k) = -\frac{\mathrm{e}^{-\mathrm{i}\pi/4}(2 + \mathrm{i}k)}{2k^2(k - \mathrm{i})^{1/2}}, \tag{†}$$

giving precise definitions of $(k - \mathrm{i})^{1/2}$ and the integration contour Γ. [*You may assume without proof that the Fourier transforms of $u(x,0)$ and of $\partial u/\partial y(x,0)$ satisfy $\bar{u}(k,0) = O\left(k^{-3/2}\right)$ and $\partial\bar{u}/\partial y(k,0) = O\left(k^{-1/2}\right)$ as $k \to \infty$.*]

Verify that (†) gives the correct expression for $u(x,0)$ when $x > 0$.

Show that, for $x < 0$,

$$u(x,0) = \frac{1}{2\pi}\int_1^\infty \frac{(2 - t)\mathrm{e}^{tx}}{t^2\sqrt{t - 1}}\,\mathrm{d}t.$$