3 Antiplane strain and torsion

3.1 Antiplane strain

The simplest two-dimensional model for elastostatics occurs when the displacement $\boldsymbol{u}$ is unidirectional, $\boldsymbol{u} = (0, 0, w)^{\mathrm{T}}$ say, and w depends only on the transverse coordinates $(x, y) = \boldsymbol{x}$. One way to create this state of *antiplane strain* is to apply a tangential traction $\sigma(\boldsymbol{x})$, in the axial direction only, to the curved boundary of a cylindrical bar, as illustrated in Figure 3.1. With the z-axis parallel to the bar, the stress tensor is

$$\tau = \begin{pmatrix} 0 & 0 & \mu\dfrac{\partial w}{\partial x} \\ 0 & 0 & \mu\dfrac{\partial w}{\partial y} \\ \mu\dfrac{\partial w}{\partial x} & \mu\dfrac{\partial w}{\partial y} & 0 \end{pmatrix}. \tag{3.1}$$

The Navier equation reduces to the two-dimensional *Laplace's equation*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w = 0, \tag{3.2}$$

to be solved inside the cross-section D of the bar, which is a region of the (x, y)-plane.

Since the bar is uniform, the unit normal $\boldsymbol{n}$ to the curved boundary is confined to the (x, y)-plane, so the applied traction is simply a shear force σ in the z-direction that is related to w by

$$\sigma(\boldsymbol{x}) = \tau_{zn} = \mu\frac{\partial w}{\partial n} \qquad \text{on} \quad \partial D. \tag{3.3}$$

The solvability condition for the elliptic Neumann problem (3.2), (3.3) is

$$\oint_{\partial D} \sigma \, \mathrm{d}s = 0. \tag{3.4}$$

Physically, this result confirms that no net traction may be applied to any cross-section.

If, instead of the traction, we were to specify the displacement w on ∂D, we would obtain the Dirichlet boundary condition

$$w = f(\boldsymbol{x}) \qquad \text{on} \quad \partial D \tag{3.5}$$

instead of (3.3). The strain energy density in antiplane strain is given by is $\mathcal{W} = \mu|\nabla w|^2$. Minimisation of the total elastic energy in the bar using the calculus of variations leads to Laplace's equation (3.2) for w, while (3.3) with $\sigma = 0$ is the natural boundary condition.

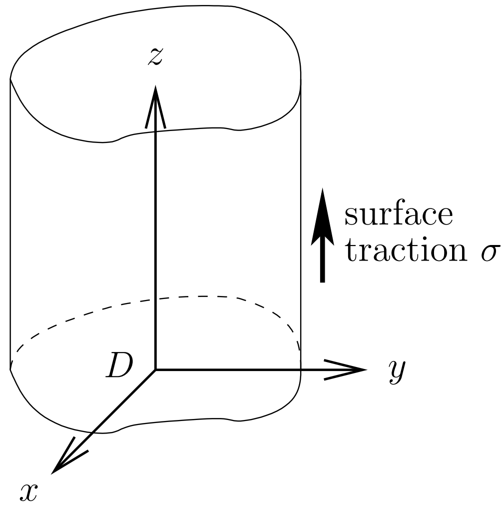

Figure 3.1: Schematic of a bar in a state of antiplane strain.

3.2 Torsion

Now consider a bar which, instead of stretching or contracting along its axis, twists under the action of moments applied at its ends. Such *torsion bars* are often used in car suspensions, and may be described by a displacement field of the form

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\Omega yz \\ \Omega xz \\ \Omega\psi(x,y) \end{pmatrix}, \tag{3.6}$$

where Ω is a constant representing the *twist* of the bar about its axis. As in §3.1, the stress tensor is of the form

$$\tau = \begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{pmatrix} \tag{3.7}$$

where, now,

$$\tau_{xz} = \mu\Omega\left(\frac{\partial\psi}{\partial x} - y\right), \qquad \tau_{yz} = \mu\Omega\left(\frac{\partial\psi}{\partial y} + x\right). \tag{3.8}$$

The Navier equation therefore implies that, as in §3.1, ψ satisfies Laplace's equation

$$\nabla^2\psi = 0 \qquad \text{in } D, \tag{3.9}$$

where D is the cross-section of the bar.

Recall that, since the bar is uniform, its unit normal $\boldsymbol{n}$ lies purely in the (x,y)-plane; if ∂D is parametrised by $x = x(s)$, $y = y(s)$, where s is arc-length, then $\boldsymbol{n} = (\mathrm{d}y/\mathrm{d}s, -\mathrm{d}x/\mathrm{d}s, 0)^{\mathrm{T}}$. Hence, assuming the curved boundary of the bar is stress-free, we must have

$$\tau_{xz}\frac{\mathrm{d}y}{\mathrm{d}s} - \tau_{yz}\frac{\mathrm{d}x}{\mathrm{d}s} = 0 \qquad \text{on } \partial D, \tag{3.10}$$

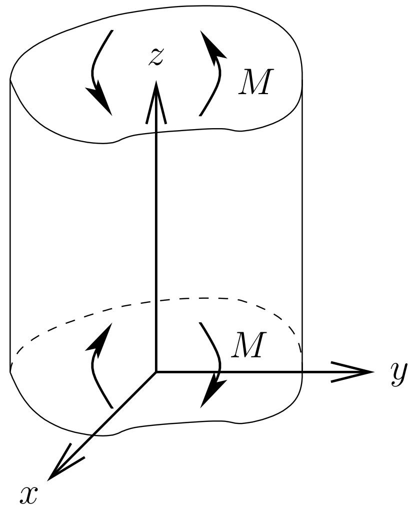

Figure 3.2: Schematic of a bar being twisted under a moment M.

and the corresponding boundary condition for ψ is

$$\frac{\partial\psi}{\partial n} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(x^2 + y^2\right) \qquad \text{on } \partial D. \tag{3.11}$$

Given the shape of the cross-section D, the displacement $w = \Omega\psi$ is uniquely determined up to an arbitrary constant, corresponding to an arbitrary uniform translation. Once ψ has been found, the moment applied at each end of the bar is given by

$$M = \iint_D (x\tau_{yz} - y\tau_{xz})\,\mathrm{d}x\mathrm{d}y = R\Omega, \tag{3.12}$$

say, where

$$R = \mu\iint_D \left\{x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x} + \left(x^2 + y^2\right)\right\}\mathrm{d}x\mathrm{d}y. \tag{3.13}$$

The constant of proportionality R between the applied moment M and the resulting twist Ω is called the *torsional rigidity* of the bar.

The simplest case occurs when D is a disc of radius a, so that the right-hand side of (3.11) is zero. It follows that ψ is constant, and then the integral in (3.13) is readily calculated to be

$$R = \frac{\pi\mu a^4}{2}. \tag{3.14}$$

When ∂D is not circular, the boundary condition (3.11) is awkward, and may be usefully reformulated as follows. Whenever the stress tensor is of the form (3.7) and there is no body force, the steady Navier equation reduces to

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0. \tag{3.15}$$

We can guarantee that (3.15) is satisfied by postulating the existence of a *stress function* $\phi(x, y)$ such that

$$\tau_{xz} = \mu\Omega\frac{\partial\phi}{\partial y}, \qquad \tau_{yz} = -\mu\Omega\frac{\partial\phi}{\partial x}, \tag{3.16}$$

where the factors of $\mu\Omega$ are introduced for later convenience. Moreover, it can be shown that the existence of ϕ is necessary as well as sufficient for solutions of (3.15) to exist, in the same way as the existence of a stream function in two-dimensional fluid dynamics guarantees mass conservation. Such *potentials* are never uniquely defined; in our case, ϕ is unique only up to the addition of an arbitrary constant.

By comparing (3.16) with (3.8), we see that ϕ and ψ are related by

$$\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} + y, \qquad \frac{\partial\psi}{\partial y} = -\frac{\partial\phi}{\partial x} - x. \tag{3.17}$$

Elimination of ϕ retrieves (3.9), while elimination of ψ reveals that ϕ satisfies *Poisson's equation*

$$\nabla^2\phi = -2 \qquad \text{in } D. \tag{3.18}$$

The zero-stress boundary condition (3.10) implies that ϕ is constant on ∂D and, without loss of generality, we may take

$$\phi = 0 \qquad \text{on} \quad \partial D. \tag{3.19}$$

The advantage of introducing the stress function ϕ is that the Neumann problem for ψ has been converted to the Dirichlet problem (3.18), (3.19), which has a unique solution; exactly the same procedure could have been applied to the antiplane strain problem (3.2), (3.3). The torsional rigidity is then found to be given in terms of ϕ by

$$R = 2\mu \iint_D \phi \, \mathrm{d}x\mathrm{d}y, \tag{3.20}$$

by using Green's theorem and (3.19).

For a circular bar of radius a, we can easily solve the Dirichlet problem (3.18), (3.19) for ϕ using plane polar coordinates (r, θ) and therefore find

$$\phi = \frac{a^2 - r^2}{2}. \tag{3.21}$$

The torsional rigidity given by (3.20) then reproduces (3.14) as expected.

3.3 Multiply-connected domains

In §3.2, we have implicitly assumed that D is simply connected. Many torsion bars are tubular in practice, and the resulting change in topology makes a big difference to the integration of our mathematical model. We now have to apply the condition (3.10) on *two* stress-free boundaries, namely the inner (∂D_i) and outer (∂D_0) surfaces of the tube, as illustrated in Figure 3.3. As before, we can deduce that ϕ must be constant on each of these boundaries, but these two constants are not necessarily equal. Hence, we may only choose ϕ to satisfy

$$\phi = 0 \qquad \text{on } \partial D_o, \qquad \phi = k \qquad \text{on } \partial D_i, \tag{3.22}$$

where k is a constant which must be determined as part of the solution.

The torsional rigidity may now be written as

$$R = 2\mu \iint_D \phi \, \mathrm{d}x\mathrm{d}y + 2\mu k A_i, \tag{3.23}$$

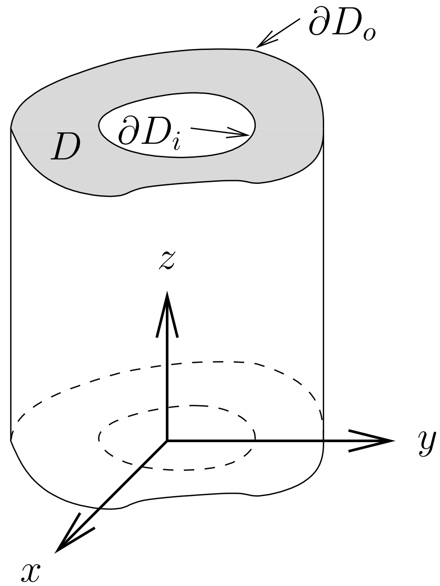

Figure 3.3: Schematic of a uniform tubular torsion bar, with inner and outer free surfaces given by ∂D_i and ∂D_o respectively.

after using Green's theorem, where A_i is the area of the hole inside ∂D_i. To obtain ϕ and thus M uniquely, we still have to evaluate the unknown constant k.

By working with the stress function ϕ, we have reduced the torsional rigidity problem to a seemingly innocuous Poisson equation (3.18) with Dirichlet boundary conditions (3.22) on the two boundaries of the annular tube. However, we have temporarily lost sight of the displacement field $w = \Omega\psi$, which has yet to be determined from (3.17). To be physically acceptable, ψ must be a single-valued function of (x, y), which implies that

$$\oint_{\partial D_i} \frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y \equiv 0. \tag{3.24}$$

By substituting for ψ in favour of ϕ, we obtain the condition

$$\oint_{\partial D_i} \frac{\partial \phi}{\partial n}\,\mathrm{d}s = -2A_i, \tag{3.25}$$

where, as before, A_i is the area of the void in the tube cross section. This constraint gives the extra information needed to determine the constant k and, hence, ϕ and M.

As an illustration, suppose that D is the circular annulus $a < r < b$ in plane polar coordinates (r, θ), as illustrated in Figure 3.4(a). Assuming that ϕ is a function of r alone, it must satisfy the boundary-value problem

$$\nabla^2\phi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -2, \qquad a < r < b, \tag{3.26a}$$

$$\phi = k, \qquad r = a, \tag{3.26b}$$

$$\phi = 0, \qquad r = b, \tag{3.26c}$$

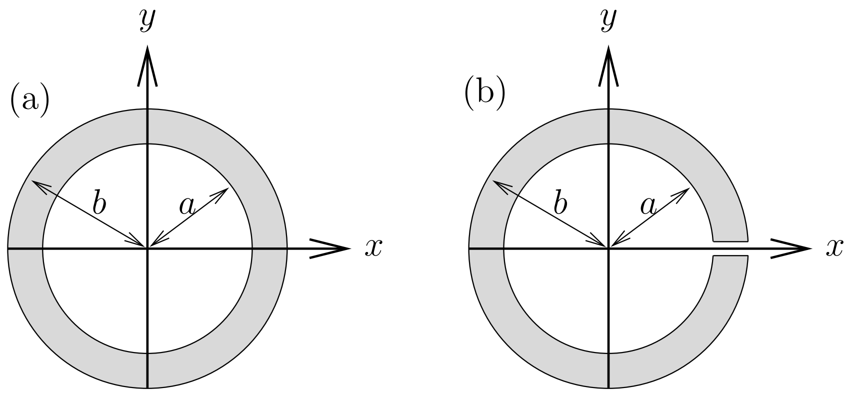

Figure 3.4: The cross-section of (a) a circular cylindrical tube; (b) a cut tube.

whose solution is easily found to be

$$\phi = \frac{b^2 - r^2}{2} + \left(k - \frac{b^2 - a^2}{2}\right) \frac{\log(r/b)}{\log(a/b)}. \tag{3.27}$$

By substituting this expression for ϕ into (3.25), we find that

$$k = \frac{b^2 - a^2}{2}, \tag{3.28}$$

which eliminates the logarithmic term from (3.27).

We therefore find that

$$\phi = \frac{b^2 - r^2}{2}, \tag{3.29}$$

and it is straightforward to substitute this into (3.23) and obtain the torsional rigidity

$$R = \frac{\mu\pi}{2}\left(b^4 - a^4\right). \tag{3.30}$$

This result can easily be confirmed by solving for ψ rather than ϕ. Even in this simple radially symmetric geometry, the bother of finding the arbitrary constant k has outweighed the convenience of introducing a stress function. When considering multiply-connected domains it is therefore often a better idea to return to the physical variable ψ.

If the tube is thin, so a and b are nearly equal, then

$$R \sim 2\mu\pi a^4 \epsilon \tag{3.31}$$

where $\epsilon = b/a - 1 \ll 1$. We can compare this with the torsional rigidity of a rusty tube, with a thin axial cut, as illustrated in Figure 3.4(b). Here the cross-section is simply connected, so that ϕ must satisfy

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -2, \qquad a < r < b, \tag{3.32a}$$

with

$$\phi = 0 \qquad \text{on} \quad r = a, b, \tag{3.32b}$$

and

$$\phi = 0 \qquad \text{on} \quad \theta = 0, 2\pi. \tag{3.32c}$$

This problem can be solved exactly by separating the variables. An arduous calculation leads to the solution

$$\phi = \frac{b^2 - r^2}{2} + \frac{\left(b^2 - a^2\right)\log\left(r/b\right)}{2\log\left(b/a\right)} + \sum_{n=1}^{\infty} C_n \cosh\left[k_n(\theta - \pi)\right] \sin\left[k_n \log\left(b/r\right)\right], \tag{3.33}$$

where

$$k_n = \frac{n\pi}{\log\left(b/a\right)}, \qquad C_n = -\frac{4\left[b^2 - (-1)^n a^2\right]\log^2\left(b/a\right)}{n\pi\cosh(k_n\pi)\left[n^2\pi^2 + 4\log^2\left(b/a\right)\right]}. \tag{3.34}$$

We can thus determine the torsional rigidity as

$$\begin{aligned} R = {} & \frac{\pi\mu\left(b^2 - a^2\right)}{2\log\left(b/a\right)}\left[\left(a^2 + b^2\right)\log\left(b/a\right) - \left(b^2 - a^2\right)\right] \\ & - \frac{16\mu\log^4\left(b/a\right)}{\pi}\sum_{n=1}^{\infty}\frac{\left[b^2 - (-1)^n a^2\right]\tanh\left[n\pi^2/\log\left(b/a\right)\right]}{\left[n^2\pi^2 + 4\log^2\left(b/a\right)\right]^2}. \end{aligned} \tag{3.35}$$

If the tube is thin, so that $b/a = 1 + \epsilon$, where $\epsilon \ll 1$, then we can expand (3.35) for small ϵ to show that

$$R \sim \frac{2\pi}{3}\mu a^4\epsilon^3. \tag{3.36}$$

Comparing (3.40) with (3.31), we see that the change in topology caused by the cut has a dramatic influence on the strength of the tube, reducing its torsional rigidity by two orders of magnitude!

We can obtain (3.36) more directly by performing the rescaling

$$r = a\left(1 + \epsilon\xi\right). \tag{3.37}$$

To lowest order in ϵ, (3.32a,b) reduces to

$$\frac{\partial^2\phi}{\partial\xi^2} \sim -2\epsilon^2 a^2, \qquad 0 < \xi < 1, \tag{3.38a}$$

with

$$\phi = 0 \qquad \text{on} \quad \xi = 0, 1, \tag{3.38b}$$

whose solution is

$$\phi = \epsilon^2 a^2 \xi(1 - \xi). \tag{3.39}$$

For a simply-connected cross-section, we use (3.20) to determine the torsional rigidity and, again using the smallness of ϵ, we obtain, up to lowest order in ϵ,

$$R = 4\pi\mu a^2\epsilon\int_0^1 \phi\,\mathrm{d}\xi = \frac{2\pi}{3}\mu a^4\epsilon^3, \tag{3.40}$$

reproducing (3.36).