Problem Sheet 0 (Initial Problem Sheet)

This problem sheet will *not* be covered in the Intercollegiate Classes for the course, but model solutions will be uploaded in due course. The purpose of this sheet is to refresh material from earlier courses that may be helpful during this course.

1. A string of linear density ρ, and length L is stretched by a constant tension T along the x-axis. The string undergoes small transverse displacements such that its position at time t is given by the graph $z = w(x,t)$.

   Show that w satisfies the one-dimensional wave equation with wave speed $c = (T/\rho)^{1/2}$.

   Show that if $x = a$ and $x = b$ are any two points along the string then

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2}\int_a^b \rho\left(\frac{\partial w}{\partial t}\right)^2\,\mathrm{d}x + \tfrac{1}{2}\int_a^b T\left(\frac{\partial w}{\partial x}\right)^2\,\mathrm{d}x\right\} = \left[T\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right]_a^b.$$

   Interpret this result in terms of the conservation of energy.

2. Consider a small line segment $\delta\boldsymbol{X}$ joining two points whose initial position vectors are $\boldsymbol{X}$ and $\boldsymbol{X}+\delta\boldsymbol{X}$. Suppose that these points are displaced to $\boldsymbol{x} = \boldsymbol{X}+\boldsymbol{u}(\boldsymbol{X})$ and $\boldsymbol{x}+\delta\boldsymbol{x} = \boldsymbol{X}+\delta\boldsymbol{X}+\boldsymbol{u}(\boldsymbol{X}+\delta\boldsymbol{X})$ respectively. Show that, for small $|\delta\boldsymbol{X}|$ and small $|\partial u_i/\partial x_j|$, the length of the displaced line element is given by

   $$|\delta\boldsymbol{x}|^2 = |\delta\boldsymbol{X}|^2 + 2e_{ij}\delta X_i\delta X_j + \text{smaller terms},$$

   $$\text{where}\quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

   is called the *linear strain tensor*.

   Define a *second-rank tensor* and show that $(\partial u_i/\partial X_j)$ and $\mathcal{E} = (e_{ij})$ are both tensors.

   Show that $e_{ij} \equiv 0$ if and only if $\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{c} + \boldsymbol{\omega}\times\boldsymbol{X}$, where $\boldsymbol{c}$ and $\boldsymbol{\omega}$ are constant vectors, and interpret this result physically.

3. Let τ_{ij} denote the x_i-component of the force per unit area exerted on a surface element whose normal is in the x_j-direction. Show that (τ_{ij}) is a tensor (the *Cauchy stress tensor*). By considering the moments acting on a small rectangle in two dimensions, or otherwise, show that $\tau_{ij} \equiv \tau_{ji}$.