Problem Sheet 3

1. An elastic beam with bending stiffness EI is in equilibrium subject to a compressive force P_0 and shear force N_0 applied at its ends, where it is clamped parallel to the x-axis. Show that, if the beam makes an angle $\theta(s)$ with the x-axis, where s is arc-length, the shear force N and bending moment M at any point satisfy

$$N = N_0 \cos\theta + P_0 \sin\theta, \qquad \frac{\mathrm{d}M}{\mathrm{d}s} - N = 0.$$

Assuming the constitutive relation $M = -EI\mathrm{d}\theta/\mathrm{d}s$, obtain the *Euler strut* equation

$$EI\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + P_0 \sin\theta + N_0 \cos\theta = 0.$$

(a) When the applied shear force is zero, obtain the dimensionless model

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \pi^2\lambda \sin\theta = 0, \qquad \theta(0) = \theta(1) = 0,$$

where the dimensionless variable ξ and parameter λ are to be defined.

(b) Assuming $|\theta| \ll 1$, show that nontrivial solutions $\theta = A\sin(n\pi\xi)$ exist when $\lambda = n^2$, where n is a positive integer.

(c) Now suppose that λ is close to one of the critical values so that $\lambda = n^2 + \varepsilon\lambda_1$, where $0 < \varepsilon \ll 1$. Show that solutions of the form

$$\theta = \varepsilon^{1/2}\left\{A_0 \sin(n\pi\xi) + \varepsilon\Theta_1 + O\left(\varepsilon^2\right)\right\}$$

exist provided the leading-order amplitude A_0 satisfies

$$A_0\left(A_0^2 - \frac{8\lambda_1}{n^2}\right) = 0.$$

Plot the resulting response diagram.

(d) Now suppose there is a *small* applied shear force N_0, so that

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \pi^2\lambda \sin\theta + \varepsilon^{3/2} F \cos\theta = 0.$$

Define F in terms of N_0. Repeat the analysis of part (c) with $n = 1$ to show that A_0 now satisfies

$$A_0\left(A_0^2 - 8\lambda_1\right) = \frac{32F}{\pi^3}.$$

Sketch the response diagram. Assuming that $F > 0$, show that a negative amplitude A_0 is possible only if the forcing parameter λ_1 exceeds $3F^{2/3}/2^{1/3}\pi^2$.

2. (a) An elastic string is stretched to a uniform tension T over a nearly flat obstacle $z = f(x)$. If a transverse body force $p(x)$ per unit length is applied, show that the transverse displacement $z = w(x)$ satisfies $T\mathrm{d}^2w/\mathrm{d}x^2 = p(x)$ in the non-contact set and $w = f$ in the contact set, with continuity of w and $\mathrm{d}w/\mathrm{d}x$ on the boundary between them.

   (b) Show that the above model is not complete by finding *three* solutions when $f(x) = -7/500$, $p(x)/T = x^2 - 4/75$ and $w = 0$ at $x = \pm 1$.

   (c) Which solution from part (b) satisfies the *complementarity conditions*

$$(w - f)\left(p - T\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) = 0, \qquad w - f \geqslant 0, \qquad p - T\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \geqslant 0? \qquad (*)$$

   Interpret these conditions physically.

3. It may be shown that $(*)$ is equivalent to the *variational inequality*

$$T\int_{-1}^{1} \frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}w}{\mathrm{d}x}\right)\mathrm{d}x \geqslant \int_{-1}^{1} p(w - v)\,\mathrm{d}x \quad \text{for all} \quad v \geqslant f. \qquad (\dagger)$$

Now we will show that $(\dagger)$ is equivalent to minimising the net strain and potential energy over all displacements that do not interpenetrate the obstacle.

   (a) Show that, if

$$U[w] = \int_{-1}^{1}\left(\frac{T}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + pw\right)\mathrm{d}x,$$

   then

$$U[w] - U[v] = \int_{-1}^{1} p\,(w - v)\;\mathrm{d}x - T\int_{-1}^{1}\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}w}{\mathrm{d}x}\right)\mathrm{d}x - \frac{T}{2}\int_{-1}^{1}\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\mathrm{d}x$$

   and deduce that, if w satisfies $(\dagger)$, then it minimises U.

   (b) Note that, if v_1 and v_2 belong to the set $\{v : v \geqslant f \text{ on } (-1, 1)\}$, then so does $\alpha v_1 + (1 - \alpha)v_2$ for $0 < \alpha < 1$ [*this means that the set is* convex]. Show that if w minimises U, then

$$U[w] \leqslant U\left[\alpha v + (1 - \alpha)w\right]$$

   for all $v \geqslant f$. Expand this inequality for small α to obtain $(\dagger)$.

4. A thin elliptical Mode III crack, whose boundary $\partial\Omega$ is given by

$$\frac{x^2}{c^2\cosh^2\varepsilon} + \frac{y^2}{c^2\sinh^2\varepsilon} = 1,$$

is subject to an antiplane strain displacement field $\boldsymbol{u} = \big(0, 0, w(x, y)\big)^{\mathrm{T}}$.

(a) If a shear stress $\tau_{yz} = \sigma$ is applied in the far field, justify the conditions

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad\qquad w \sim \frac{\sigma y}{\mu} \quad \text{as } x^2 + y^2 \to \infty.$$

(b) Show that the *Joukowsky transformation*

$$x + \mathrm{i}y = z = \frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

conformally maps the region $|\zeta| > \mathrm{e}^{\varepsilon}$ ($\varepsilon > 0$) onto the outside of the crack. What happens as $\varepsilon \to 0$? What is the inverse map from z to ζ?

(c) Introducing polar coordinates (r, θ) such that $\zeta = r\mathrm{e}^{\mathrm{i}\theta}$, show that w satisfies the conditions

$$\frac{\partial w}{\partial r} = 0 \quad \text{on } r = \mathrm{e}^{\varepsilon}, \qquad\qquad w \sim \frac{c\sigma}{2\mu} r\sin\theta \quad \text{as } r \to \infty.$$

Hence obtain the solution

$$w = \frac{c\sigma}{2\mu}\,\mathrm{Im}\left\{\zeta - \frac{\mathrm{e}^{2\varepsilon}}{\zeta}\right\}.$$

(d) In the limit $\varepsilon \to 0$, deduce that

$$w = \frac{\sigma}{\mu}\,\mathrm{Im}\left\{\sqrt{z^2 - c^2}\right\}, \tag{‡}$$

and carefully define the square root.

5. If the displacement in antiplane strain is given by $w(x, y) = \mathrm{Im}\{f(z)\}$, where $z = x + \mathrm{i}y$, show that the corresponding stress components are

$$\tau_{xz} = \mu\,\mathrm{Im}\{f'(z)\}, \qquad\qquad \tau_{yz} = \mu\,\mathrm{Re}\{f'(z)\}.$$

Hence show that the stress components ahead of the crack, on $y = 0,\ x > c$, due to the displacement field (‡), are given by

$$\tau_{xz} = 0, \qquad\qquad \tau_{yz} = \frac{\sigma x}{\sqrt{x^2 - c^2}}.$$

Suppose that the crack tip propagates when the *stress intensity factor*

$$K_{\mathrm{III}} = \sqrt{2\pi}\lim_{x\downarrow c}\left\{\tau_{yz}(x, 0)\sqrt{x - c}\right\}$$

exceeds a critical value $K_\star$. Deduce that the crack will grow if the applied shear stress exceeds $K_\star/\sqrt{\pi c}$.