Problem Sheet 4

1. In a two-dimensional granular material, show that the normal stress N and shear stress F on a line element with unit normal $\boldsymbol{n} = (\cos\theta, \sin\theta)^{\mathrm{T}}$ lie on the *Mohr circle*

$$F^2 + \left(N - \frac{1}{2}(\tau_{xx} + \tau_{yy})\right)^2 = \frac{(\tau_{xx} - \tau_{yy})^2}{4} + \tau_{xy}^2.$$

Sketch the Mohr circle in the (N, F)-plane and explain why the Coulomb yield condition $|F| = -N\tan\phi$ is satisfied at exactly one value of θ if and only if

$$-(\tau_{xx} + \tau_{yy})\sin\phi = \sqrt{(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2}.$$

If there is no body force and negligible inertia, deduce that the Airy stress function satisfies

$$\frac{\partial^2 \mathfrak{A}}{\partial x^2}\frac{\partial^2 \mathfrak{A}}{\partial y^2} - \left(\frac{\partial^2 \mathfrak{A}}{\partial x \partial y}\right)^2 = \frac{\cos^2\phi}{4}\left(\nabla^2 \mathfrak{A}\right)^2. \qquad (*)$$

By differentiating with respect to x and y, write $(*)$ as a first-order system for $p = \partial^2\mathfrak{A}/\partial x^2$ and $q = \partial^2\mathfrak{A}/\partial y^2$. Show that the system is hyperbolic.

2. Show that

$$w = \frac{b\theta}{2\pi} = \frac{b}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \qquad (\dagger)$$

is a possible displacement in equilibrium antiplane strain, and evaluate the corresponding stress components. What would you have to do to a cylinder of metal for it to adopt this configuration?

3. Show that the Tresca yield criterion in plane strain leads to the condition

$$\sqrt{\frac{1}{4}\left(\tau_{xx} - \tau_{yy}\right)^2 + \tau_{xy}^2} \leqslant \tau_Y,$$

where τ_Y is the yield stress. Assuming that inertia and gravity are negligible, deduce that the Airy stress function in a yielding material satisfies the equation

$$\left(\nabla^2\mathfrak{A}\right)^2 + 4\left\{\left(\frac{\partial^2 \mathfrak{A}}{\partial x \partial y}\right)^2 - \frac{\partial^2 \mathfrak{A}}{\partial x^2}\frac{\partial^2 \mathfrak{A}}{\partial y^2}\right\} = 4\tau_Y^2. \qquad (\ddagger)$$

By differentiating with respect to x and y (or otherwise), show that $(\ddagger)$ is hyperbolic, with characteristics satisfying

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\frac{\partial^2 \mathfrak{A}}{\partial x \partial y} \pm \tau_Y\right) \Big/ \left(\frac{\partial^2 \mathfrak{A}}{\partial x^2} - \frac{\partial^2 \mathfrak{A}}{\partial y^2}\right).$$

4. Consider a circular torsion bar of radius a and shear modulus μ subject to a twist $\Omega > \Omega_c = \tau_Y/\mu a$, where τ_Y is the yield stress. Show that the bar yields in a region $s < r < a$, where $s = \tau_Y/\mu\Omega$, and that the stress components in the bar are given by

$$\tau_{xz} = \begin{cases} -\mu\Omega y & 0 \leqslant r < s, \\ -\mu\Omega ys/r & s < r < a, \end{cases} \qquad \tau_{yz} = \begin{cases} \mu\Omega x & 0 \leqslant r < s, \\ \mu\Omega xs/r & s < r < a. \end{cases}$$

Hence show that the torque M exerted on the bar satisfies

$$\frac{2M}{\pi a^3 \tau_Y} = \frac{1}{3}\left(4 - \frac{\Omega_c^3}{\Omega^3}\right).$$

Now suppose that, after a maximum twist Ω_{M} has been applied, the torque is gradually released. Assuming that the material instantaneously reverts to being elastic, show that the stress components are now given by

$$\begin{aligned} \tau_{xz} &= \begin{cases} -\mu\Omega y & 0 \leqslant r < s_{\mathrm{M}}, \\ \mu\left(\Omega_{\mathrm{M}} - \Omega\right) y - \mu\Omega_{\mathrm{M}} s_{\mathrm{M}} y/r & s_{\mathrm{M}} < r < a, \end{cases} \\ \tau_{yz} &= \begin{cases} \mu\Omega x & 0 \leqslant r < s_{\mathrm{M}}, \\ -\mu\left(\Omega_{\mathrm{M}} - \Omega\right) x + \mu\Omega_{\mathrm{M}} s_{\mathrm{M}} x/r & s_{\mathrm{M}} < r < a, \end{cases} \end{aligned} \qquad (\S)$$

where $s_{\mathrm{M}} = \tau_Y/\Omega_{\mathrm{M}}\mu$ is the maximum radius of the yielded region. Hence show that the corresponding torque is given by

$$\frac{2M}{\pi a^3 \tau_Y} = \frac{\Omega - \Omega_{\mathrm{M}}}{\Omega_c} + \frac{1}{3}\left(4 - \frac{\Omega_c^3}{\Omega_{\mathrm{M}}^3}\right),$$

and evaluate the residual twist Ω_0 that remains when the torque is completely released.

Suppose we now twist the bar in the opposite direction, so that M is negative. Use (§) to show that the bar yields again at $r = a$ when

$$\Omega = \Omega_{\mathrm{M}} - 2\Omega_c, \qquad \frac{2M}{\pi a^3 \tau_Y} = -\frac{2}{3} - \frac{\Omega_c^3}{3\Omega_{\mathrm{M}}^3} > -1.$$

[*Thus, after the bar has yielded in one direction, a smaller torque is required to make it yield in the opposite direction. This is known as the* Bauschinger effect.]

5. Consider plane strain outside a circular hole $r = a$ which is inflated to a pressure P that is greater than the yield stress τ_Y. Show that the material yields in a region $a < r < s$, where

$$s(P) = a \exp\left(\frac{P}{2\tau_Y} - \frac{1}{2}\right).$$

Now suppose that P increases to a maximum value P_{M} before decreasing back to zero. Assuming that the material instantaneously becomes elastic when $P < P_{\mathrm{M}}$, show that there is a residual stress field

$$\tau_{rr} = \begin{cases} -P_{\mathrm{M}} + 2\tau_Y \log\left(\frac{r}{a}\right) + \frac{a^2 P_{\mathrm{M}}}{r^2} & a < r < s_{\mathrm{M}}, \\ -\frac{\tau_Y s_{\mathrm{M}}^2}{r^2} + \frac{a^2 P_{\mathrm{M}}}{r^2} & s_{\mathrm{M}} < r < \infty, \end{cases}$$

$$\tau_{\theta\theta} = \begin{cases} -P_{\mathrm{M}} + 2\tau_Y + 2\tau_Y \log\left(\frac{r}{a}\right) - \frac{a^2 P_{\mathrm{M}}}{r^2} & a < r < s_{\mathrm{M}}, \\ \frac{\tau_Y s_{\mathrm{M}}^2}{r^2} - \frac{a^2 P_{\mathrm{M}}}{r^2} & s_{\mathrm{M}} < r < \infty, \end{cases}$$

where $s_{\mathrm{M}} = s(P_{\mathrm{M}})$. Deduce that the material will yield again at $r = a$ as the pressure is released if $P_{\mathrm{M}} > 2\tau_Y$.