Problem Sheet 4: Solutions

1. The two-dimensional stress tensor is

$$\mathcal{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix}.$$

The normal stress on a line element with normal $\boldsymbol{n}$ is

$$N = \boldsymbol{n}\cdot(\mathcal{T}\boldsymbol{n}) = (\cos\theta, \sin\theta)\begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \end{pmatrix} = \tau_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \tau_{yy}\sin^2\theta,$$

which can also be written

$$N = \tfrac{1}{2}(\tau_{xx} + \tau_{yy}) + \tfrac{1}{2}(\tau_{xx} - \tau_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta. \tag{1}$$

The shear stress is

$$S = \boldsymbol{t}\cdot(\mathcal{T}\boldsymbol{n}) = (-\sin\theta, \cos\theta)\begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \end{pmatrix} = (\tau_{yy} - \tau_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta),$$

which gives

$$S = -\tfrac{1}{2}(\tau_{xx} - \tau_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta. \tag{2}$$

The two relationships (1) and (2) defined a circle (the Mohr circle) in the (N, S)-plane that is parametrized by θ. It is easy to see by direct substitution that

$$S^2 + \left[N - \tfrac{1}{2}(\tau_{xx} + \tau_{yy})\right]^2 = \tfrac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2.$$

This is a circle radius $\mathcal{R} = \left[(\tau_{xx} - \tau_{yy})^2/4 + \tau_{xy}^2\right]^{1/2}$ and centre $((\tau_{xx} + \tau_{yy})/2, 0)$. Since a granular material cannot withstand any tensile stress, we need $N \le 0\ \forall\theta$, which in turn requires that

$$0 \ge -\frac{\tau_{xx} + \tau_{yy}}{2} + \left[\tfrac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2\right]^{1/2}$$

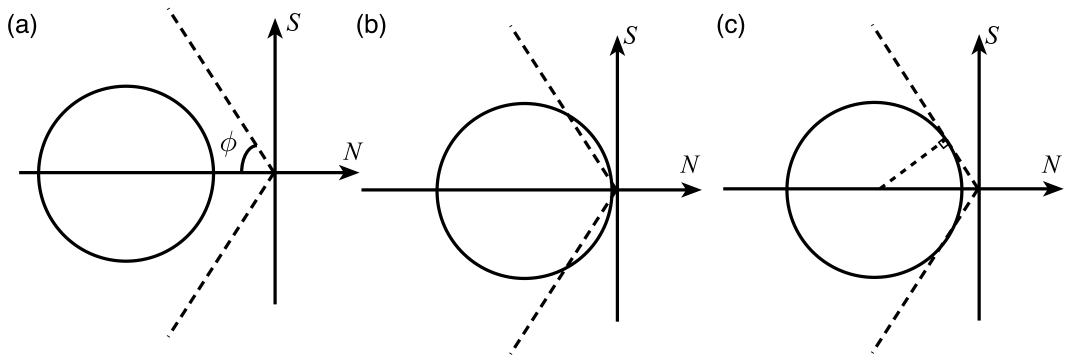
or

$$\frac{(\tau_{xx} + \tau_{yy})^2}{4} \ge \tfrac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2, \tag{3}$$

which is equivalent to $\tau_{xx}\tau_{yy} \ge \tau_{xy}^2$; $\tau_{xx} \le 0$, $\tau_{yy} \le 0$, so that the stress tensor $\mathcal{T}$, is negative semi-definite.

The Coulomb condition requires that $|S| \le -N \tan\phi$ with ϕ the angle of friction and equality when the material yields.

There are then three cases of interest:



(a) $|S| < -N \tan\phi$ for all θ; the material has not yielded anywhere.

(b) For some values of θ, $|S| > -N \tan\phi$. This is not possible (it violates the Coulomb friction assumption).

(c) There is precisely one value of θ for which $|S| = -N \tan\phi$, so the material yields.

In this last case, trigonometry gives

$$\sin\phi = \frac{\left[\frac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2\right]^{1/2}}{-\frac{\tau_{xx}+\tau_{yy}}{2}}.$$

Rearranging, we find that

$$(\tau_{xx} + \tau_{yy})^2 \sin^2\phi = (\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2$$

or

$$0 = (\tau_{xx} + \tau_{yy})^2(1 - \sin^2\phi) + 4(\tau_{xy}^2 - \tau_{xx}\tau_{yy}).$$

Finally, we have

$$\tau_{xx}\tau_{yy} - \tau_{xy}^2 = \frac{\cos^2\phi}{4}(\tau_{xx} + \tau_{yy})^2.$$

With no body force, the two-dimensional Navier equation is

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}.$$

As in lectures, we assume that any motion (including plastic flow) is slow enough that the acceleration terms may be neglected. We then introduce an Airy stress function $\mathcal{A}(x, y)$ such that $\tau_{xx} = \partial^2\mathcal{A}/\partial y^2$, $\tau_{yy} = \partial^2\mathcal{A}/\partial x^2$, $\tau_{xy} = -\partial^2\mathcal{A}/\partial x\partial y$, and hence the Navier equation is automatically satisfied.

Substituting the stress components associated with the stress function into (3), we find that

$$\frac{\cos^2\phi}{4}(\nabla^2\mathcal{A})^2 = \frac{\partial^2\mathcal{A}}{\partial x^2}\frac{\partial^2\mathcal{A}}{\partial y^2} - \left(\frac{\partial^2\mathcal{A}}{\partial x\partial y}\right)^2, \tag{4}$$

as required.

To show that this equation is hyperbolic, we let

$$p = \frac{\partial^2\mathcal{A}}{\partial x^2}, \quad q = \frac{\partial^2\mathcal{A}}{\partial y^2}, \quad r = \frac{\partial^2\mathcal{A}}{\partial x\partial y}$$

so that symmetry of mixed partial derivatives gives

$$\frac{\partial r}{\partial x} = \frac{\partial p}{\partial y}, \quad \frac{\partial r}{\partial y} = \frac{\partial q}{\partial x}.$$

The condition (4) may be written

$$\frac{\cos^2\phi}{4}(p+q)^2 = pq - r^2,$$

and, differentiating with respect to x and y we find

$$q\frac{\partial p}{\partial x} + p\frac{\partial q}{\partial x} - 2r\frac{\partial p}{\partial y} = \frac{\cos^2\phi}{2}(p+q)\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x}\right)$$

and

$$q\frac{\partial p}{\partial y} + p\frac{\partial q}{\partial y} - 2r\frac{\partial q}{\partial x} = \frac{\cos^2\phi}{2}(p+q)\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial y}\right),$$

respectively.

This can be written as a system of linear equations in $(\partial p/\partial x, \partial q/\partial x)^T$ and $(\partial p/\partial y, \partial q/\partial y)^T$, which has characteristics $\lambda = \mathrm{d}y/\mathrm{d}x$ where λ satisfies

$$\det\begin{pmatrix} -2r - \lambda\left[q - \frac{\cos^2\phi}{2}(p+q)\right] & -\lambda\left[p - \frac{\cos^2\phi}{2}(p+q)\right] \\ q - \frac{\cos^2\phi}{2}(p+q) & p - \frac{\cos^2\phi}{2}(p+q) + 2\lambda r \end{pmatrix} = 0, \tag{5}$$

which can be expanded out to give

$$\left[q - \tfrac{\cos^2\phi}{2}(p+q)\right]\lambda^2 + 2r\lambda + p - \tfrac{\cos^2\phi}{2}(p+q) = 0.$$

The discriminant is $\Delta = r^2 - \left[p - \frac{\cos^2\phi}{2}(p+q)^2\right]\left[q - \frac{\cos^2\phi}{2}(p+q)^2\right]$, which simplifies to $\Delta = r^2 - pq + \frac{\cos^2\phi}{2}(p+q)^2 - \frac{\cos^4\phi}{4}(p+q)^2$. However, since we know that $pq - r^2 = \cos^2\phi(p+q)^2/4$ from (4) we can therefore write

$$\Delta = \tfrac{\cos^2\phi}{4}(p+q)^2(1-\cos^2\phi) = \frac{\sin^2(2\phi)}{16}(p+q)^2 > 0.$$

Hence, the PDE is hyperbolic, as required.

Finally, we can note that the gradients of the characteristics are

$$\lambda = \frac{-2r \pm \Delta^{1/2}}{2q - (p+q)\cos^2\phi}.$$

2. In antiplane strain at equilibrium, we have $\boldsymbol{u} = w(x,y)\boldsymbol{k}$ and $\nabla^2 w = 0$. In plane polars (r,θ) Laplace's equation becomes

$$0 = \nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0.$$

Hence $w(r,\theta) = b\theta/(2\pi)$ is a possible solution.

From the lecture notes, the stress components in antiplane strain (expressed in polar coordinates) all vanish except

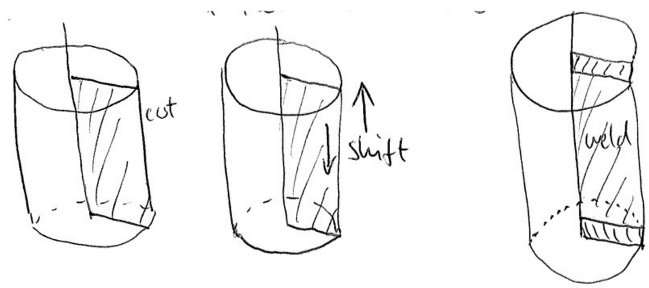
$$\tau_{rz} = \mu\frac{\partial w}{\partial r}, \quad \tau_{\theta z} = \frac{\mu}{r}\frac{\partial w}{\partial \theta}.$$

Here the only nonzero stress is

$$\tau_{\theta z} = \frac{\mu b}{2\pi r}.$$

Note that this stress is singular as $r \to 0$, where θ is not well-defined in any case.

This displacement could be realized by cutting a cylinder along a ray from the axis, then shifting one face up a distance b and welding/gluing the two face back together.



[*Note:* After this 'cut-and-weld' operation, the cylinder would be in a state of self-stress: $\tau_{\theta z} \neq 0$ despite no external forces being imposed on the boundary.

The compatibility condition in antiplane strain comes from $\partial^2 w/\partial x \partial y = \partial^2 w/\partial y \partial x$ which immediately leads to

$$\frac{\partial \tau_{yz}}{\partial x} = \frac{\partial \tau_{xz}}{\partial y}.$$

Here

$$\tau_{xz} = -\frac{\mu b}{2\pi} \frac{y}{x^2 + y^2}, \quad \tau_{yz} = \frac{\mu b}{2\pi} \frac{x}{x^2 + y^2},$$

which satisfies the compatibility condition except at $(x, y) = (0, 0)$, where the stresses are not defined.

In fact, it can be shown that

$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = \mu b \, \delta(x) \delta(y)$$

where $\delta(x)$ is the usual Dirac δ-function. As a result, this displacement field corresponds to a line of incompatibility along the z-axis.]

3. From Q. 1, the shear stress on a line element with normal $\boldsymbol{n} = (\cos\theta, \sin\theta, 0)^T$ is

$$\begin{aligned} F &= -\tfrac{1}{2}(\tau_{xx} - \tau_{yy}) \sin(2\theta) + \tau_{xy} \cos(2\theta) \\ &= \left[\tfrac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2\right]^{1/2} \sin(2\theta - \alpha) \end{aligned}$$

for some α. Hence the maximum is

$$\max_{\theta} |F| = \left[\tfrac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2\right]^{1/2} \leq \tau_Y$$

under the Tresca condition, with equality when the material is yielding.

The equality (at yielding) can be rearranged to give

$$(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2 = 4\tau_Y^2.$$

When inertia and gravity are negligible, the 2-D Navier equations

$$\begin{aligned} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} &= 0, \end{aligned}$$

imply the existence of an Airy stress function $\mathcal{A}$ such that $\tau_{xx} = \partial^2\mathcal{A}/\partial y^2$, $\tau_{yy} = \partial^2\mathcal{A}/\partial x^2$, $\tau_{xy} = -\partial^2\mathcal{A}/\partial x\partial y$. The Tresca condition (at yield) then immediately gives

$$(\nabla^2\mathcal{A})^2 + 4\left[\left(\frac{\partial^2\mathcal{A}}{\partial x\partial y}\right)^2 - \frac{\partial^2\mathcal{A}}{\partial x^2}\frac{\partial^2\mathcal{A}}{\partial y^2}\right] = 4\tau_Y^2,$$

as required.

To show that this equation is hyperbolic, we first let

$$p = \frac{\partial^2\mathcal{A}}{\partial x^2}, \quad q = \frac{\partial^2\mathcal{A}}{\partial y^2}, \quad r = \frac{\partial^2\mathcal{A}}{\partial x\partial y}$$

so that symmetry of mixed partial derivatives gives

$$\frac{\partial r}{\partial x} = \frac{\partial p}{\partial y}, \quad \frac{\partial r}{\partial y} = \frac{\partial q}{\partial x}$$

while the Tresca yielding condition may be written

$$(p-q)^2 + 4r^2 = 4\tau_Y^2.$$

Differentiating with respect to x and y we find that

$$\begin{aligned} 0 &= 2(p-q)\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial x}\right) + 8r\frac{\partial p}{\partial y} \\ 0 &= 2(p-q)\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial y}\right) + 8r\frac{\partial q}{\partial x}. \end{aligned}$$

Writing this as a linear system, we find that the system has characteristics $\mathrm{d}y/\mathrm{d}x = \lambda$ where

$$\det\begin{pmatrix} 4r + \lambda(q-p) & \lambda(p-q) \\ p-q & q-p-4\lambda r \end{pmatrix} = 0, \tag{6}$$

which gives that

$$(p-q)\lambda^2 - 4r\lambda - (p-q) = 0,$$

which has solution

$$\lambda = (p-q)^{-1}\left[2r \pm \sqrt{4r^2 + (p-q)^2}\right].$$

However, from the original PDE, we know that $4r^2 + (p-q)^2 = 4\tau_Y^2$, so that

$$\lambda = \frac{2r \pm 2\tau_Y}{p-q}.$$

Since these solutions are real, the original problem is hyperbolic, with characteristics

$$\lambda = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\left(\frac{\partial^2 \mathcal{A}}{\partial x \partial y} \pm \tau_Y\right)}{\frac{\partial^2 \mathcal{A}}{\partial x^2} - \frac{\partial^2 \mathcal{A}}{\partial y^2}}.$$

4. Recall (e.g. from Sheet 1, Question 5) that the displacement in an unyielded circular bar is $\boldsymbol{u} = \Omega(-yz, xz, 0)^T$, which corresponds to non-zero stress components $\tau_{xz} = -\mu\Omega y$ and $\tau_{yz} = \mu\Omega x$. The applied torque is given by

$$M = \int\int_D (x\tau_{yz} - y\tau_{xz})\ \mathrm{d}x\mathrm{d}y = \mu\Omega \int\int_D (x^2 + y^2)\ \mathrm{d}x\mathrm{d}y$$

where, here, the cross-section D is the circle $x^2 + y^2 < a^2$.

Use polar coordinates (r, θ) to show that

$$M = \mu\Omega \int_0^{2\pi} \int_0^a r^2 \cdot r\mathrm{d}r\mathrm{d}\theta = \frac{\pi a^4 \mu\Omega}{2}.$$

The Tresca condition states that $\tau_{xz}^2 + \tau_{yz}^2 \leq \tau_Y^2$ where τ_Y is the yield stress. Here, this becomes

$$\mu^2\Omega^2(x^2 + y^2) \leq \tau_Y^2,$$

or $\mu\Omega r \leq \tau_Y$.

This is first violated at $r = a$ when

$$\Omega = \Omega_c = \frac{\tau_Y}{\mu a}.$$

For $\Omega > \Omega_c$, there must be a plastic region near the boundary of the bar, denoted $s < r < a$.

In $r < s$, the bar is still elastic and we have the same solution as before: in $r < s$, $\tau_{xz} = -\mu\Omega y$ and $\tau_{yz} = \mu\Omega x$.

At the free boundary, $r = s$, we must have that the yield criterion is satisfied, i.e.

$$\tau_{xz}^2 + \tau_{yz}^2 = \tau_Y^2 \implies \mu\Omega s = \tau_Y \implies s = \frac{\tau_Y}{\mu\Omega}$$

or $s/a = \Omega_c/\Omega$. (Note that s starts at a when $\Omega = \Omega_c$ and decreases towards the centre of the bar as Ω increases.)

In the plastic region $s < r < a$, the Tresca condition becomes an equality: $\tau_{xz}^2 + \tau_{yz}^2 = \tau_Y^2$ to be solved along with the Navier equation

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.$$

To solve this, we introduce a stress function $\phi(x, y)$ such that $\tau_{xz} = \partial\phi/\partial y$, $\tau_{yz} = -\partial\phi/\partial x$ so that the Tresca condition becomes

$$|\nabla\phi| = \tau_Y.$$

Since ϕ is only defined up to a constant (this domain is simply connected), we can choose $\phi = 0$ on $r = a$.

We then seek an axisymmetric solution, $\phi = \phi(r)$, so that $\mathrm{d}\phi/\mathrm{d}r = \pm\tau_Y$. To choose the sign, we note that in $r < s$ $\mathrm{d}\phi/\mathrm{d}r = -\mu\Omega r < 0$; hence, continuity of $\mathrm{d}\phi/\mathrm{d}r$ across $r = s$ shows that we require $\mathrm{d}\phi/\mathrm{d}r < 0$. We find that

$$\phi = \tau_Y(a - r)$$

and hence

$$\frac{\partial\phi}{\partial x} = -\frac{\tau_Y x}{r}, \quad \frac{\partial\phi}{\partial y} = -\frac{\tau_Y y}{r}.$$

We therefore have that

$$\tau_{xz} = -\frac{\tau_Y x}{r}, \quad \tau_{yz} = \frac{\tau_Y x}{r} \quad \text{in} \quad s < r < a$$

so that the applied torque

$$M = \int\int_D (x\tau_{yz} - y\tau_{xz})\ \mathrm{d}x\mathrm{d}y = 2\pi\int_0^2 \mu\Omega r^2 \cdot r\ \mathrm{d}r + 2\pi\int_s^a \tau_Y r \cdot r\ \mathrm{d}r$$

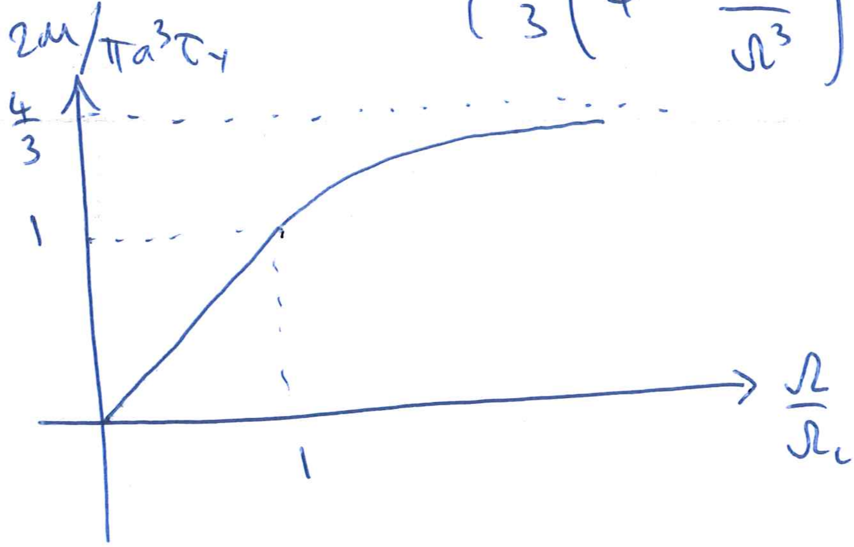
or

$$\frac{2M}{\pi a^3 \tau_Y} = \tfrac{1}{3}\left(4 - \frac{\Omega_c^3}{\Omega^3}\right), \quad \text{for} \quad \Omega > \Omega_c.$$

[Note that the elastic solution obtained for $\Omega \leq \Omega_c$ can be written as

$$\frac{2M}{\pi a^3 \tau_Y} = \frac{2}{\pi a^3 \tau_Y} \cdot \frac{\pi a^4 \mu \Omega}{2} = \frac{a \mu \Omega}{\tau_Y} = \frac{\Omega}{\Omega_c}.$$

So we have
$$\frac{2M}{\pi a^3 \tau_Y} = \begin{cases} \Omega/\Omega_c & \Omega \leq \Omega_c \\ \frac{1}{3}\left(4 - \frac{\Omega_c^3}{\Omega^3}\right) & \Omega > \Omega_c \end{cases}$$



The bar becomes easier to twist after it has yielded, and fails completely at a finite torque $\frac{2M}{\pi a^3 \tau_Y} = \frac{4}{3}$.]

When $\Omega = \Omega_M$, we have $s = s_M = \dfrac{a\,\Omega_c}{\Omega_M}$

and $$\tau_{xz} = \begin{cases} -\mu\,\Omega_M\, y & r < s_M \\ -\dfrac{\tau_Y\, y}{r} & r > s_M \end{cases}$$

$$\tau_{yz} = \begin{cases} \mu\,\Omega_M\, x & r < s_M \\ \dfrac{\tau_Y\, x}{r} & r > s_M \end{cases}$$

Denote the subsequent twist by $\hat{\Omega} < 0$ so $\Omega = \Omega_M + \hat{\Omega}$.
We assume the additional response is purely elastic, so gives rise to an additional stress

$$\hat{\tau}_{xz} = -\mu\,\hat{\Omega}\, y \qquad \hat{\tau}_{yz} = \mu\,\hat{\Omega}\, x$$

Just add these to the initial stress when $\hat{\Omega} = 0$ to get

$$\tau_{xz} = \begin{cases} -\mu\,\Omega_M\, y - \mu\,\hat{\Omega}\, y & r < s_M \\ -\dfrac{\tau_Y\, y}{r} - \mu\,\hat{\Omega}\, y & r > s_M \end{cases}$$

$$\tau_{yz} = \begin{cases} \mu\,\Omega_M\, x + \mu\,\hat{\Omega}\, x & r < s_M \\ \dfrac{\tau_Y\, x}{r} + \mu\,\hat{\Omega}\, x & r > s_M \end{cases}$$

A simple rearrangement gives

$$\tau_{xz} = \begin{cases} -\mu\Omega y & r < s_m \\ \mu(\Omega_m - \Omega)y - \mu\Omega_m s_m \dfrac{y}{r} & r > s_m \end{cases}$$

$$\tau_{yz} = \begin{cases} \mu\Omega x & r < s_m \\ \mu(\Omega - \Omega_m)x + \mu\Omega_m s_m \dfrac{x}{r} & r > s_m \end{cases}$$

as required. Now the torque is given by

$$M = \iint_D (x\tau_{yz} - y\tau_{xz})\,dx\,dy$$

$$= 2\pi\int_0^{s_m} \mu\Omega r^2 . r\,dr + 2\pi\int_{s_m}^{a}\left[\mu(\Omega-\Omega_m)r^2 + \mu\Omega_m s_m r\right] r\,dr$$

$$= \frac{\pi\mu\Omega s_m^4}{2} + \frac{\pi\mu(\Omega-\Omega_m)}{2}(a^4 - s_m^4) + \frac{2\pi\mu\Omega_m s_m}{3}(a^3 - s_m^3)$$

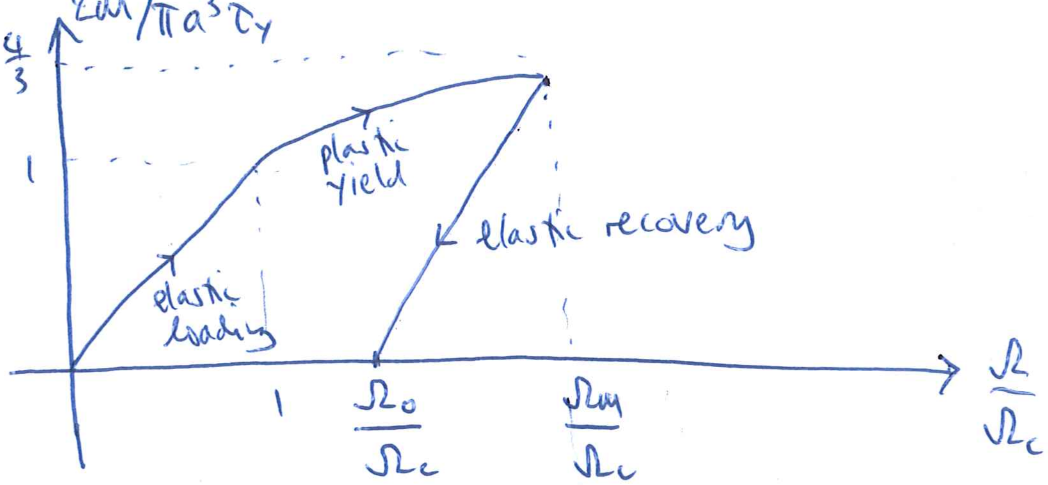
$$\Rightarrow \frac{2M}{\pi a^3\tau_Y} = \frac{\mu\Omega}{\tau_Y}\frac{s_m^4}{a^3} + \frac{\mu(\Omega-\Omega_m)a}{\tau_Y}\left(1 - \frac{s_m^4}{a^4}\right) + \frac{4}{3}\frac{\mu\Omega_m s_m}{\tau_Y}\left(1 - \frac{s_m^3}{a^3}\right)$$

$$= \frac{\Omega}{\Omega_c}\left(\frac{\Omega_c^4}{\Omega_m^4}\right) + \left(\frac{\Omega}{\Omega_c} - \frac{\Omega_m}{\Omega_c}\right)\left(1 - \frac{\Omega_c^4}{\Omega_m^4}\right) + \frac{4}{3}\left(1 - \frac{\Omega_c^3}{\Omega_m^3}\right)$$

i.e. $$\boxed{\frac{2M}{\pi a^3\tau_Y} = \frac{\Omega - \Omega_m}{\Omega_c} + \frac{1}{3}\left(4 - \frac{\Omega_c^3}{\Omega_m^3}\right)}$$

[The second term is the initial torque when $\Omega = \Omega_M$;

the first term is a linear elastic response when $\Omega < \Omega_M$:



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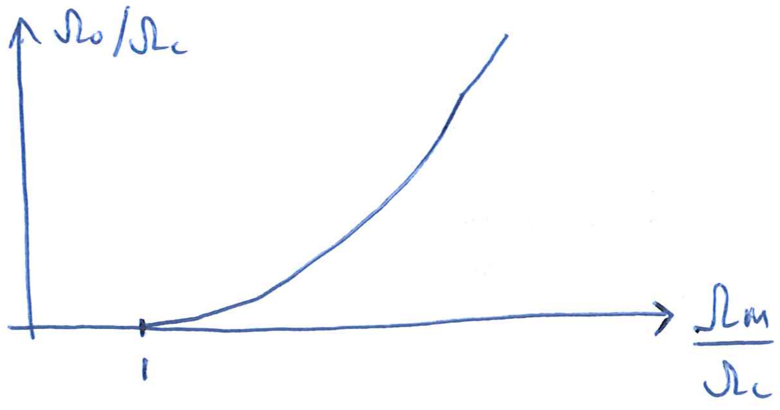
When $M=0$, we get a residual twist Ω_0 given by

$$\frac{\Omega_0}{\Omega_c} = \frac{\Omega_M}{\Omega_c} - \frac{1}{3}\left(4 - \frac{\Omega_c^3}{\Omega_M^3}\right)$$

[This is zero when $\Omega_M = \Omega_c$, i.e. when we haven't twisted the bar enough to make it yield. Then Ω_0 increases as Ω_M increases past Ω_c:



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From (*) we have

$$\tau_{xz}^2 + \tau_{yz}^2 = \begin{cases} \mu^2 \Omega^2 r^2 & r < s_m \\ \left[\mu(\Omega_m - \Omega) - \mu\Omega_m \frac{s_m}{r}\right]^2 r^2 & r > s_m \end{cases}$$

ie.

$$\tau_{xz}^2 + \tau_{yz}^2 = \begin{cases} \mu^2 \Omega^2 r^2 & r < s_m \\ \mu^2\left[\Omega_m (r - s_m) - \Omega r\right]^2 & r > s_m \end{cases}$$

You can easily verify that this is an increasing function of r, so takes its maximum value at $r = a$, where

$$\tau_{xz}^2 + \tau_{yz}^2 = \mu^2 \left[(\Omega_m - \Omega) a - \Omega_m s_m\right]^2$$

The bar will yield again when this equals τ_Y^2, ie.

$$\mu\left[(\Omega_m - \Omega) a - \Omega_m s_m\right] = \tau_Y = \mu a \Omega_c$$

i.e. $\Omega_m - \Omega - \Omega_c = \Omega_c$ [recall $\Omega_m s_m = a\Omega_c$]

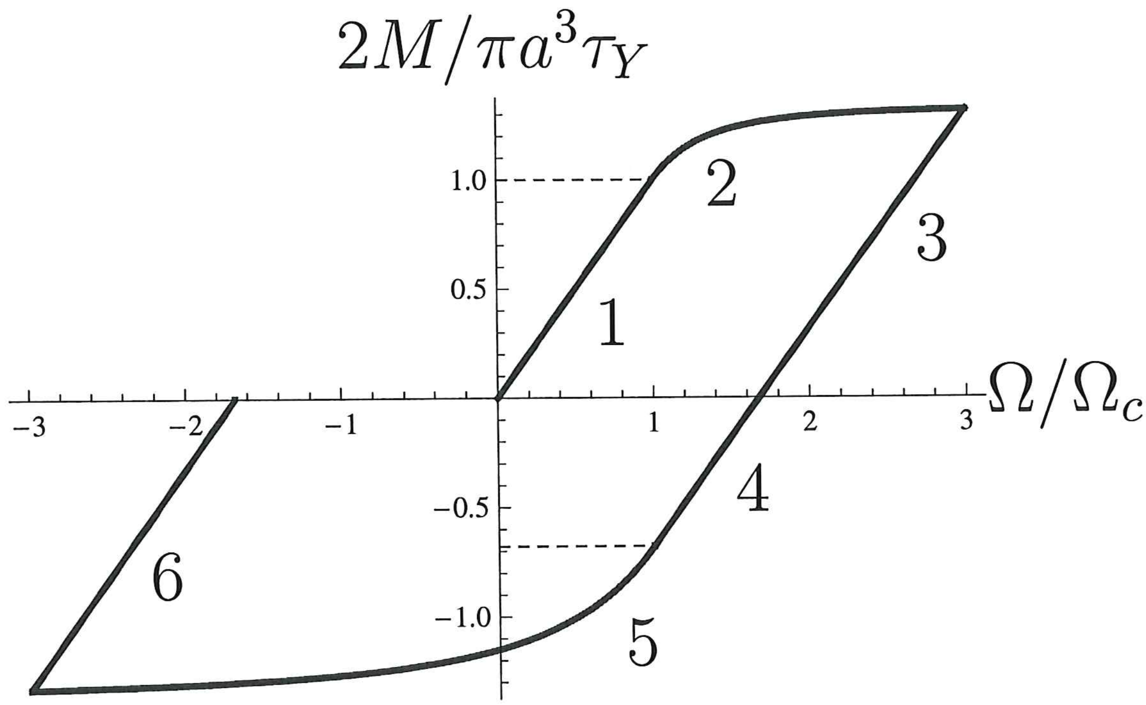
$$\Rightarrow \boxed{\Omega = \Omega_m - 2\Omega_c}$$

Substitute into expression for M:

$$\frac{2M}{\pi a^3 \tau_Y} = -\frac{2\Omega_c}{\Omega_c} + \frac{1}{3}\left(4 - \frac{\Omega_c^3}{\Omega_m^3}\right) = -\frac{2}{3} - \frac{\Omega_c^3}{3\Omega_m^3}.$$

as required.

[Here's the torque-versus-twist behaviour during a complete loading cycle:]



1. The bar deforms elastically
2. Yield occurs and plastic deformation commences
3. The bar recovers elastically as the load is released.
4. Now a torque is applied in the reverse direction
5. The bar yields at a <u>lower</u> critical torque.
6. Finally the load is released and a permanent negative twist remains.

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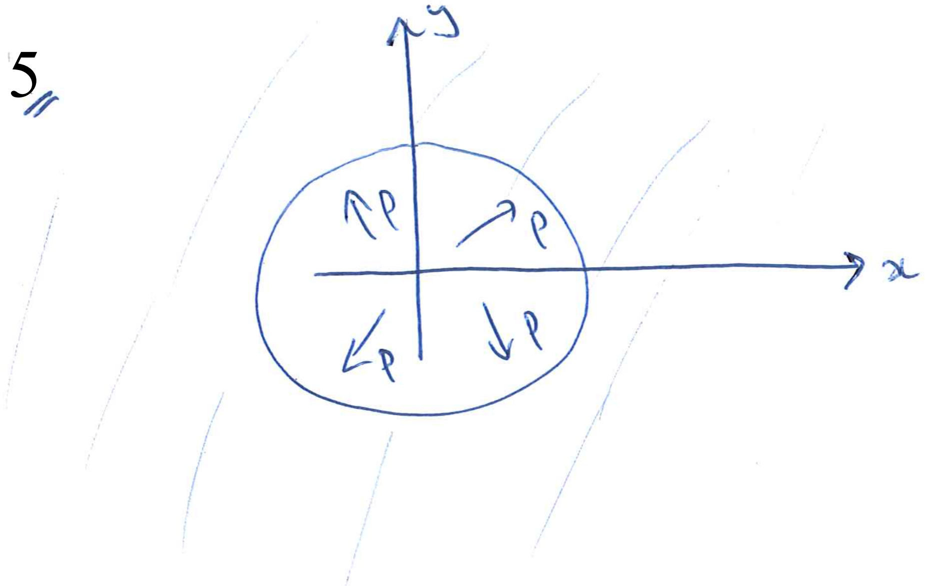


Use plane polars (r,θ) and look for a radially symmetric solution with $\tau_{r\theta}=0$, τ_{rr} and $\tau_{\theta\theta}$ functions of r only.

The Navier equation is $\quad \dfrac{d\tau_{rr}}{dr} + \dfrac{\tau_{rr}-\tau_{\theta\theta}}{r} = 0$

and the boundary conditions are $\quad \tau_{rr} = -P$ on $r=a$

$\tau_{rr}, \tau_{\theta\theta} \to 0$ as $r\to\infty$.

While the material remains elastic, we can use the constitutive relations

$$\tau_{rr} = (\lambda+2\mu)\frac{du_r}{dr} + \lambda\frac{u_r}{r}$$

$$\tau_{\theta\theta} = \lambda\frac{du_r}{dr} + (\lambda+2\mu)\frac{u_r}{r}$$

where $\underset{\sim}{u} = u_r(r)\,\underset{\sim}{e}_r$ is the displacement.

Hence the Navier equation becomes

$$(\lambda+2\mu)\frac{d^2u_r}{dr^2} + \frac{\lambda}{r}\frac{du_r}{dr} - \frac{\lambda u_r}{r^2} + 2\mu\left(\frac{du_r}{dr} - \frac{u_r}{r}\right) = 0$$

i.e. $$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

Try $u_r = r^m$ $\Rightarrow$ $m(m-1) + m - 1 = 0$

$\Rightarrow$ $(m+1)(m-1) = 0 \Rightarrow m = \pm 1$

So general solution is $u_r = Ar + \frac{B}{r}$

$$\Rightarrow \tau_{rr} = (\lambda+2\mu)\left(A - \frac{B}{r^2}\right) + \lambda\left(A + \frac{B}{r^2}\right) = 2(\lambda+\mu)A - \frac{2\mu B}{r^2}$$

$$\tau_{\theta\theta} = \lambda\left(A - \frac{B}{r^2}\right) + (\lambda+2\mu)\left(A + \frac{B}{r^2}\right) = 2(\lambda+\mu)A + \frac{2\mu B}{r^2}$$

We want $\tau_{rr}, \tau_{\theta\theta} \to 0$ as $r \to \infty$ $\Rightarrow$ $\underline{A = 0}$.

$\tau_{rr} = -P$ on $r = a$ $\Rightarrow$ $\frac{2\mu B}{a^2} = P$

so that $$\boxed{\tau_{rr} = -\frac{Pa^2}{r^2}, \quad \tau_{\theta\theta} = \frac{Pa^2}{r^2}}$$

The Tresca condition in polars is

$$\sqrt{\frac{1}{4}(\tau_{rr} - \tau_{\theta\theta})^2 + \tau_{r\theta}^2} \leq \tau_Y$$

i.e. $|\tau_{rr} - \tau_{\theta\theta}| \leq 2\tau_Y$ when $\tau_{r\theta} = 0$

with equality when the material is yielding.

Here $\tau_{\theta\theta} > \tau_{rr}$ so $|\tau_{rr} - \tau_{\theta\theta}| = \tau_{\theta\theta} - \tau_{rr}$

and Tresca becomes $\dfrac{2Pa^2}{r^2} \leq 2\tau_Y$.

This first fails at $r = a$ when $\underline{P = \tau_Y}$.

For $P > \tau_Y$, there will be a plastic region near the hole, say in $a < r < s$.

In $r > s$ the material is still elastic so we still have

$$\tau_{rr} = -\frac{2\mu B}{r^2}, \quad \tau_{\theta\theta} = \frac{2\mu B}{r^2}.$$

The Tresca condition on the plastic boundary $r = s$ gives

$$\tau_{\theta\theta} - \tau_{rr} = \frac{4\mu B}{s^2} = 2\tau_Y \Rightarrow B = \frac{s^2 \tau_Y}{2\mu}$$

$$\Rightarrow \boxed{\tau_{rr} = -\frac{s^2\tau_Y}{r^2}, \quad \tau_{\theta\theta} = \frac{s^2\tau_Y}{r^2} \quad \text{in } r > s}$$

In $r < s$ we solve the Navier equation

$$\frac{d\tau_{rr}}{dr} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0$$

and the Tresca condition $\tau_{\theta\theta} - \tau_{rr} = 2\tau_Y$ simultaneously.

Hence $\quad \left.\begin{aligned} \tau_{rr} &= 2\tau_Y \log r + C \\ \tau_{\theta\theta} &= 2\tau_Y + 2\tau_Y \log r + C \end{aligned}\right\}$ in $r<s$.

The normal stress τ_{rr} must be continuous at $r=s$:

$$2\tau_Y \log s + C = -\tau_Y$$

and the bc on $r=a$ is $\tau_{rr} = -P$ so

$$2\tau_Y \log a + C = -P$$

Hence $\quad C = -P - 2\tau_Y \log a \quad$ so that

$$\boxed{\left.\begin{aligned} \tau_{rr} &= -P + 2\tau_Y \log\left(\frac{r}{a}\right) \\ \tau_{\theta\theta} &= 2\tau_Y - P + 2\tau_Y \log\left(\frac{r}{a}\right) \end{aligned}\right\} \text{ in } r<s}$$

and $\quad 2\tau_Y \log s = -\tau_Y - C = P - \tau_Y + 2\tau_Y \log a$

$$\Rightarrow \quad \log\left(\frac{s}{a}\right) = \frac{P}{2\tau_Y} - \frac{1}{2}$$

$$\Rightarrow \quad \boxed{s = a \exp\left(\frac{P}{2\tau_Y} - \frac{1}{2}\right)}$$

[Notice that $s=a$ when $P = \tau_Y$; then s increases rapidly as P is increased past the yield stress.]

Now when the stress is released, the material becomes elastic so we just superimpose an elastic stress on the stress field just obtained:

$$\tau_{rr} = \begin{cases} -\dfrac{S_m^2 \tau_Y}{r^2} - \dfrac{2\mu B}{r^2} & r > S_m \\ -P_m + 2\tau_Y \log\left(\dfrac{r}{a}\right) - \dfrac{2\mu B}{r^2} & r < S_m \end{cases}$$

$$\tau_{\theta\theta} = \begin{cases} \dfrac{S_m^2 \tau_Y}{r^2} + \dfrac{2\mu B}{r^2} & r > S_m \\ 2\tau_Y - P_m + 2\tau_Y \log\left(\dfrac{r}{a}\right) + \dfrac{2\mu B}{r^2} & r < S_m. \end{cases}$$

On $r=a$ we have $\tau_{rr} = -P$, so

$$-P = -P_m - \frac{2\mu B}{a^2} \Rightarrow \underline{2\mu B = a^2(P - P_m).}$$

So when the applied pressure P reaches zero, $B = -\dfrac{a^2 P_m}{2\mu}$

and hence the residual stress is

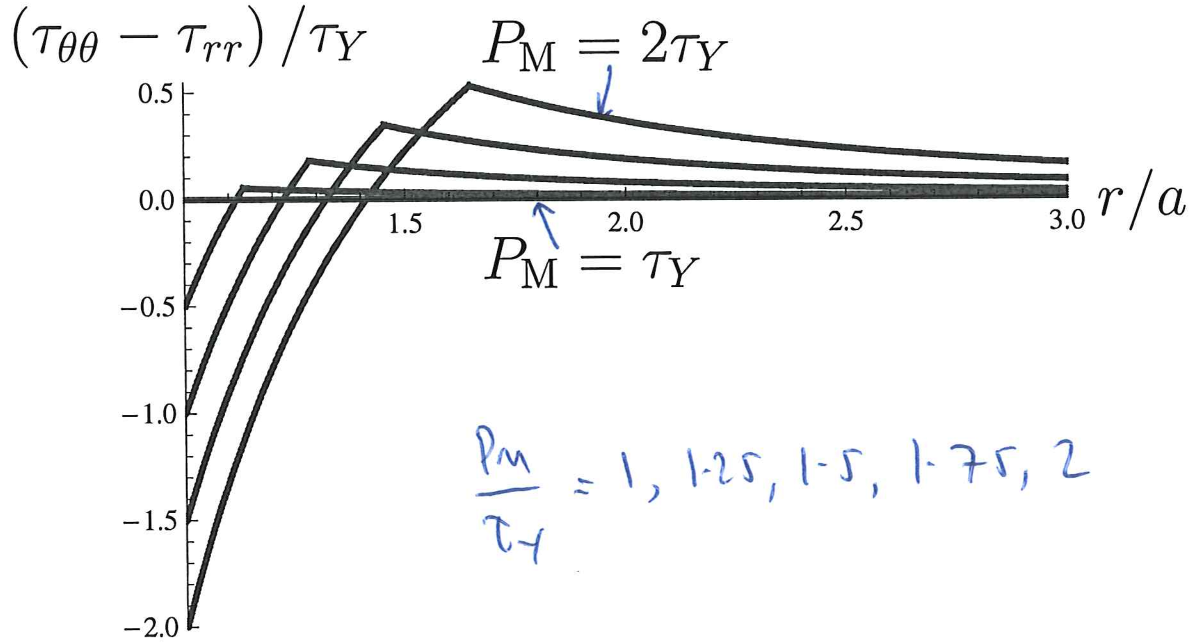
$$\tau_{rr} = \begin{cases} -\dfrac{S_m^2 \tau_Y}{r^2} + \dfrac{a^2 P_m}{r^2} & r > S_m \\ -P_m + 2\tau_Y \log\left(\dfrac{r}{a}\right) + \dfrac{a^2 P_m}{r^2} & r < S_m \end{cases}$$

$$\tau_{\theta\theta} = \begin{cases} \dfrac{S_m^2 \tau_Y}{r^2} - \dfrac{a^2 P_m}{r^2} & r > S_m \\ 2\tau_Y - P_m + 2\tau_Y \log\left(\dfrac{r}{a}\right) - \dfrac{a^2 P_m}{r^2} & r < S_m \end{cases}$$

Hence

$$\tau_{\theta\theta} - \tau_{rr} = \begin{cases} \dfrac{2\left(s_M^2 \tau_Y - a^2 P_M\right)}{r^2} & r > s_M \\[2ex] 2\tau_Y - \dfrac{2a^2 P_M}{r^2} & r < s_M \end{cases}$$

where $s_M = a \exp\left(\dfrac{P_M}{2\tau_Y} - \dfrac{1}{2} \right)$



Here we see how the residual stress increases as the maximum applied pressure P_M increases past τ_Y. Near the hole $r=a$, the material is under a <u>negative</u> shear stress. When $P_M = 2\tau_Y$, we get $|\tau_{\theta\theta} - \tau_{rr}| = 2\tau_Y$ at $r=a$, so the material yields again as it recovers.