Problem sheet 4

1) Monotone Operators

(1) Let $A : M \to X^*$ be a monotone operator,

i.e. • monotone: $\langle A(u) - A(v), u - v \rangle \geq 0$

• hemicontinuous: $t \mapsto \langle A(tu + (1-t)v), w \rangle$ cont. $\forall u, w, v$ ..

Claim 1: A satisfies cond. (H3), i.e.

(i) If $u_n \rightharpoonup u$, $A(u_n) \overset{w}{\rightharpoonup} \xi$ in X^*

then $\langle \xi, u \rangle \leq \underline{\lim} \langle A(u_n), u_n \rangle$

(ii) Equality in (i) implies

$\langle A(u) - \xi, u - v \rangle \leq 0 \quad \forall v \in M.$

By Minty (ii) is equivalent to

$\langle A(v) - \xi, u - v \rangle \leq 0 \quad \forall v \in M$

Proof of Claim 1:

Let $u_n \overset{w}{\rightharpoonup} u$ (always $u_n \in M$)

s.t. $A(u_n) \overset{w^*}{\rightharpoonup} \xi$ (w^* is eq. to w as X reflexive)

Then $\langle A(u_n) - \xi, v \rangle \to 0 \quad \forall v \in X$

in particular

$\langle A(u_n) - \xi, u \rangle \to 0$

and also $\langle A(u), u_n - u \rangle \to 0$ as $u_n \rightharpoonup u$

So:

(i) $\langle A(u_n), u_n - u \rangle = \underbrace{\langle A(u_n) - A(u), u_n - u \rangle}_{\geq 0} + \underbrace{\langle A(u), u_n - u \rangle}_{\to 0}$

giving

$\underline{\lim} \langle A(u_n), u_n \rangle \geq \overline{\lim} \langle A(u_n), u \rangle = \langle \xi, u \rangle$

so (i) ✓

$\widetilde{(ii)}$ Assume we have

$\underline{\lim} \langle A(u_n), u_n \rangle = \langle \xi, u \rangle$

We claim that

$\langle A(v) - \xi, u - v \rangle \leq 0 \quad \forall v \in M$

So let $v \in M$ any element

Monotone $\Rightarrow$ $\langle A(v) - A(u_n), u_n - v \rangle \leq 0$

so

$$0 \geq \overline{\lim} \Big(\underbrace{\langle A(v), u_n \rangle}_{\to \langle A(v), u\rangle \text{ by w conv of } u_n} - \langle A(u_n), u_n \rangle - \underbrace{\langle A(v), v \rangle}_{\text{fixed}} + \underbrace{\langle A(u_n), v \rangle}_{\to \langle \xi, v\rangle \text{ by w conv. of } A(u_n)} \Big)$$

$$= \langle A(v), u \rangle - \underbrace{\underline{\lim} \langle A(u_n), u_n \rangle}_{= \langle \xi, u \rangle \text{ by ass.}} - \langle A(v), v \rangle + \langle \xi, v \rangle$$

$$= \langle A(v) - \xi, u - v \rangle \quad \to \boxed{III}\,(ii) \to \boxed{VII}\,(i)$$

(2) Assume $u_{1,2}$ are solutions of

$$\langle A(u_i), u_i - v \rangle \leq 0 \quad \forall v \in M$$

Then

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle = \langle A(u_1), u_1 - u_2 \rangle + \langle A(u_2), u_2 - u_1 \rangle \leq 0 \quad \text{so} \quad u_1 = u_2$$

(3) Know: (H3) satisfied

so show

Claim 1: (H1) holds, i.e. A maps bd. sets to bd sets] holds by ass.

Claim 2: (H2) holds, i.e. A is coercive wrt. some $u_0 \in M$, $\frac{\langle A(u), u - u_0 \rangle}{\|u - u_0\|} \to \infty$ $\|u\| \to \infty$

Then Thm from lecture implies $\exists$ FP. and by (2) FP is unique

~~Proof of (H1)~~:

~~Let R>0.~~

Proof of (H2):

Fix any $u_0 \in M$.

Then $\langle A(u), u - u_0 \rangle$

$$= \langle A(u) - A(u_0), u - u_0 \rangle + \langle A(u_0), u - u_0 \rangle$$

$$\geq c \cdot \|u - u_0\|^2 - \frac{C}{\sqrt{c}} \cdot \|u - u_0\| \sqrt{c} \quad \text{as } A(u_0) \in X^*$$

$$\overset{C.S.}{\geq} \left(c - \frac{c}{2}\right) \|u - u_0\|^2 - \left(\frac{C}{\sqrt{c}}\right)^2 \cdot \frac{1}{2}$$

so

$$\frac{\langle A(u), u - u_0 \rangle}{\|u - u_0\|} \geq \frac{c}{2} \|u - u_0\| - \frac{\tilde{C}}{\|u - u_0\|} \xrightarrow[\|u - u_0\| \to \infty]{} \infty.$$

2) **Monotonicity $\Leftrightarrow$ Convexity:**

F Gateâux diff.

so $\forall x \in X \ \exists \ F'(x) \in X^*$

with $F'(x)(v) = \partial_v F(u) \ \forall v \in X$

F convex $\Rightarrow F': X \to X^*$ monotone:

- F convex $\Rightarrow \forall u,v \in X \ \ t \mapsto g_{u,v}(t) := F(tu + (1-t)v)$ is convex
- F Gateau-diff. $\Rightarrow$ directional der. exists so $g_{u,v}$ is different. with $g_{u,v}'(t) = \langle F'(tu+(1-t)v), (u-v) \rangle$
- g convex $\Rightarrow$ $g_{u,v}'$ non-decreasing

so $0 \leq g_{u,v}'(1) - g_{u,v}'(0) = \langle F'(u), u-v \rangle - \langle F'(v), u-v \rangle$

so F' monotone.

$F': X \to X^*$ monotone $\Rightarrow$ F convex

Let $u,v \in X$, $g = g_{u,v}$ as above

F' monotone $\Rightarrow$ $g'(s) - g'(t) \geq 0$ for $s \geq t$

Since:

$$g'(s) - g'(t) = \langle F'(\underbrace{su + (1-s)v}_{=: u_s}), u-v \rangle - \langle F'(u_s + (t-s)(u-v)), u-v \rangle$$

$$= \frac{1}{s-t} \cdot \underbrace{\langle F'(u_s) - F'(u_s - (s-t)(u-v)), (s-t)(u-v) \rangle}_{\geq 0 \text{ as } F' \text{ monotone.}}$$

so g' non-decreasing

so g convex and hence

$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v) \quad \forall t \in [0,1] \ \forall u,v$

so F convex.

Q3 Strongly monotone operator on $H^2(-1,1)$.

a) • $A: X \to X^*$ is linear and bounded,

$$\|A(u)\|_{X^*} = \sup_{\|v\|_X=1} |A(u)| \leq \|u''\|_{L^2(\Omega)} \leq \|u\|_X$$

so continuous so also hemicontinuous

• As $H^2(\Omega) \hookrightarrow C^1(\Omega)$ and $u \in H_0^1(\Omega)$ we have

• by "H_0^1-vers" of Poincaré ineq.

$$\int u^2 \, dx \leq c_1 \int |u'|^2 dx$$

• by MV=0 version of Poincaré ineq.

$$\|u'\|_{L^2}^2 \leq c_2 \|u''\|_{L^2}^2$$

since $\int_\Omega u' dx = u(1) - u(-1) = 0$

so

$$\|u\|_{H^2(\Omega)}^2 \leq (1+c_1)\|u'\|_{L^2}^2 + \|u''\|_{L^2}^2 \leq \underbrace{(c_2(1+c_1)+1)}_{=:\frac{1}{c_0}} \underbrace{\|u''\|_{L^2}^2}_{=\langle A(u),u\rangle}$$

as this holds $\forall u \in X$ and as A is linear thus also

$$\langle A(u) - A(v), u-v\rangle = \langle A(u-v), u-v\rangle \geq c_0 \cdot \|u-v\|_{H^2}^2.$$

b) • Let $B_0(u)(v) = u(0)v(0)$.

Then $B_0: X \to X^*$ well def., linear & bounded so continuous as $H^2(\Omega) \hookrightarrow C^0(\Omega)$

so $|u(0)| \leq c_3 \|u\|_{H^2}$

• Map $f: v \mapsto \int_\Omega x \cdot v(x) \, dx$ is an element of X^* as • $|\langle f, v\rangle| \leq \|x\|_{L^2} \cdot \|v\|_{L^2} \leq \|v\|_{H^2}$

• linear.

so $B: u \longmapsto B_0(u) + f$ is well def.

also

$$|\langle B(u) - B(v), u-v\rangle| = |\langle B_0(u-v), u-v\rangle| \leq c_3^2 \|u-v\|_{H^2}^2$$

So for $\mu_0 > 0$ chosen s.t. $c_0 - \mu_0 c_3 = c_1 > 0$

we have that $\forall |\mu| \leq \mu_0$

$$< F_\mu(u-v), u-v > \geq \underbrace{(c_0 - \mu_0 c_3^2)}_{=c_1} \|u-v\|_{H^2}^2$$

so strongly monotone.

As A, B_0 map bd sets of X to bd sets of X^* and as f is ind. of u F_μ thus satisfies all ass. of Q1c)

so $\exists!\, u$ s.t. $< F_\mu(u), u-v > \leq 0 \quad \forall v \in X$

Given any w, choosing $v = u \pm w$ gives

$< F_\mu(u), w > = 0 \quad \forall w \in X$

so $F_\mu(u) = 0$

c) $\mu > 0$

$$< F_\mu(u), v > = \int u'' \cdot v'' dx + \mu(u(0) v(0)) + \mu \cdot \int x \cdot v(x) dx$$

Claim:

$F_\mu(u) = 0 \iff u$ is minimiser of

$$I_\mu(v) := \frac{1}{2} \int |v''|^2 dx + \frac{\mu}{2} (v(0))^2 + \mu \cdot \int x \cdot v(x) dx$$

Proof:

"$\Leftarrow$" u minimiser $\Rightarrow \frac{d}{dt}\big|_{t=0} I_\mu(u+tv) = 0 \quad \forall v$

and this differential is

$< F_\mu(u), v >$

"$\Rightarrow$" Let v be any element of X, u s.t. $F_\mu(u) = 0$

$\to$ need to show $I_\mu(v) - I_\mu(u) \geq 0$

REASON WHY THIS HOLDS: I_μ convex:

$$I_\mu(v) - I_\mu(u) = \frac{1}{2} \int |v''|^2 - |u''|^2 + \frac{\mu}{2} [(v(0))^2 - (u(0))^2]$$

$$+ \underbrace{\mu \int x (v-u)(x) dx}_{= -< A(u), v-u >}$$

$$- \mu \cdot u(0) (v(0) - u(0))$$

so $I_\mu(v) - I_\mu(u)$

$$= \frac{1}{2} \int |v''|^2 - 2u' \cdot v' + |u''|^2$$

$$+ \frac{\mu}{2} \left[v(0)^2 - 2v(0)u(0) + u(0)^2 \right]$$

$$= \frac{1}{2} \| (v-u)'' \|^2 + \frac{\mu}{2} \cdot (v-u)(0)^2$$

≥ 0 with "=" iff $u = v$

Remark:
actually this also holds for $\mu \leq 0$,
provided $|\mu| \leq \mu_0$, μ_0 as in b)

as we get

$$2 I_\mu(v) - I_\mu(u) = \langle A(u-v), u-v \rangle$$
$$+ \mu \langle B(u-v), u-v \rangle$$

which we know is > 0 for ...

Q4

$I(v) = \int |\Delta v|^2 + f \cdot v \, dx \quad , f \in L^2(\Omega)$

(1) E-L eq

weak form of EL eq.

$\frac{d}{dt} I(u+tw) = \boxed{2\int \Delta u \cdot \Delta w + \int f \cdot w \overset{!}{=} 0}$

$= \int (2\Delta^2 u + f)\, w \overset{!}{=} 0 \quad \forall w$

so EL : (strong form of) $\boxed{\Delta^2 u + 2f = 0}$

Assume u is a critical point so

$\int 2\Delta u \cdot \Delta w + f \cdot w = 0 \quad \forall w \in H_0^1(\Omega) \cap H^2(\Omega)$

Then if v is s.t. $u - v \in H_0^1(\Omega)$, $v \in H^2(\Omega)$

then

$I(v) - I(u) = \int |\Delta v|^2 - |\Delta u|^2 + \int f(v-u)$

$= \int |\Delta v|^2 - |\Delta u|^2 - 2\Delta u(\Delta v - u)$

$= \|\Delta(u-v)\|_{L^2}^2 \geq 0$

with "=" iff $\Delta u = \Delta v$

As also $u = v$ on $\partial\Omega$ (in sense of traces)

weak max princ. implies that

$u = v$ on Ω (a.e.)

[Alternative would be to use hint of next part for this last bit]

(2) Let $g \in C^2(\mathbb{R}^n)$, $g \leq 0$ on $\partial\Omega$

$M = \{v \in H^2(\Omega) \cap H_0^1(\Omega) : v \geq g \text{ on } \Omega\}$

<u>Claim</u>: $\exists$ unique minimiser of I on M

<u>Proof</u>:

Let $X = H_0^1(\Omega) \cap H^2(\Omega)$ with $\boxed{\|u\|_X^2 = \|u\|_{H_0^1}^2 + \|\Delta u\|_{L^2}^2}$ which is a reflexive seperable Banach space.

Let M as in Q which is convex, closed also NON-EMPTY. and as $g \leq 0$ on $\partial\Omega$

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Define

$A: M \to X^*$

$u \mapsto A(u)$, $A(u)(w) = 2\int \Delta u \cdot \Delta w + \int f \cdot w$

Claim: A is a strongly monotone operator. and maps bd sets to bd sets

Proof: • $A(u) = B(u) + F$

for • $F(v) = \int f \cdot v \, dx$

• $B(u)(v) = \int \Delta u \cdot \Delta v \, dx$

as F, $B(u)$ are linear and bounded,

$\| F(v) \| \leq \|f\|_{L^2} \cdot \|v\|_{L^2} \leq \|f\|_{L^2} \cdot \|v\|_X$

$\| B(u)(v) \| \leq \|u\|_{H^2} \cdot \|v\|_{H^2}$

we have $A : M \to X^*$

and as $u \mapsto B(u)$ is linear and bounded, $\|B(u)\|_{X^*} \leq \|u\|_{H^2}$ to bd sets.

A is continuous & maps bounded sets to bd sets.

• $\langle A(u) - A(v), u - v \rangle = \|\Delta(u-v)\|_{L^2}^2$

$$\overset{\text{Hint}}{\geq} \frac{1}{2c^2} \|(u-v)\|_{H^1}^2 + \frac{1}{2} \|\Delta(u-v)\|_{L^2}^2$$

$$\geq c_0 \cdot \|u - v\|_X^2 .$$

so exist. of A UNIQUE SOLUTION to follows from Q1.

$\langle A(u), u - v \rangle \leq 0 \quad (*)$

As for every $v \in M$

$$I(v) - I(u) = \int |\Delta v - \Delta u|^2 + \langle A(u), v - u \rangle$$

$$\geq 0 \quad \text{"="only if } \Delta u = \Delta v$$

u is a minimiser of I on M

and finally if v was any other minimiser we'd have that v is also a sol. of $(*)$ so must be $\equiv u$.

Proof that M is NON-EMPTY:

Let u be unique sol. of

$$\begin{cases} -\Delta u = -\Delta g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(exists as $\Delta g \in C^{\infty}(\mathbb{R}^n)$ so $\in C^2(\bar\Omega)$).

Then $-\Delta(u-g) = 0$, $u \geq g$ on $\partial\Omega$

so $u \geq g$ a.e. by weak MP.

Q5) $\Omega = B_1(0) \subset \mathbb{R}^2$

$$-\Delta u + u^5 = 1 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation:

As $H_0^1(\Omega) \underset{\&\text{ comp.}}{\overset{\text{cont.}}{\hookrightarrow}} L^p(\Omega) \ \forall\, p < \infty$

have in particular $u^5 \in L^2(\Omega)$ if $u \in H_0^1$

so weak-form.: Find $u \in H_0^1(\Omega)$ s.t.

$$\int \nabla u \cdot \nabla v \, dx + \int u^5 \cdot v \, dx = \int v \, dx \quad \forall v \in H_0^1(\Omega)$$

is welldef.

Reformulate as FP problem:

Let $T(u) := (-\Delta)^{-1}(1 - u^5)$

Then $T : H_0^1(\Omega) \xrightarrow[\substack{\text{cont.} \\ \text{by lemma}}]{u \mapsto 1-u^5 = f(u)} L^2(\Omega) \xrightarrow[\text{comp}]{(-\Delta)^{-1}} H_0^1(\Omega)$

- Simple subsol./supersol.

$\overline{u} \equiv 1$ then $-\Delta \overline{u} = 0 = f(\overline{u})$

and $\overline{u} \geq 0$ on $\partial\Omega$

$\underline{u} = \cancel{-1}\ 0$ then $-\Delta \underline{u} = 0 \leq f(\underline{u}) = 1$

and $\underline{u} \leq 0$.

- For $f_\lambda(u) = 1 - u^5 + \lambda u$ to be $\nearrow$ on $\cancel{\ \ }$ $[0,1]$

choose e.g. $\lambda = 5$ then $f_\lambda'(u) = 5 - 5u^4 \geq 0$.

Constructive method:

Define iteratively

$u_0 = \underline{u}$

$u_j = T_\lambda(u_{j-1})$ where $T_\lambda(u) = (-\Delta + \lambda)^{-1}(f_\lambda(u))$

<u>Claim</u>: (i) $\cancel{\ \ }$ $u_j \geq u_{j-1}$

(ii) $u_j \leq \overline{u}$

Proof of (i):

For $j=1$ have

- $(-\Delta+\lambda)(u_1) = f_\lambda(u_0) = f_\lambda(\underline{u})$

  $\geq (-\Delta+\lambda)(\underline{u})$

  as $\underline{u}$ subsol.

- $u_1 = 0 \geq \underline{u}$ on $\partial\Omega$

so, as weak MP holds also for $-\Delta+\lambda$, $\lambda \geq 0$

$u_1 \geq u_2$

- For $j \geq 1$ assume claim true for j

Then

$(-\Delta+\lambda)(u_{j+1}) = f_\lambda(u_j) \geq f_\lambda(u_{j-1})$ ($u_j \geq u_{j-1}$, f_λ monot.)

$= (-\Delta+\lambda)(u_j)$

$u_{j+1} = u_j$ on $\partial\Omega$

So claim true again by wMP.

(ii)

Finally: Know $u_0 \leq \bar{u}$

If $u_j \leq \bar{u}$ then get

$(-\Delta+\lambda)(u_{j+1}) = f_\lambda(u_j) \leq f_\lambda(\bar{u}) \leq (-\Delta+\lambda)\bar{u}$

$u_{j+1} = 0 \leq \bar{u}$ on $\partial\Omega$

so by wMP

$u_{j+1} \leq \bar{u}$

So have monotone sequence

$\underline{u} \leq u_j \leq u_{j+1} \leq \bar{u}$

~~so (u_j) bounded in $L^\infty(\Omega)$~~

~~so (u_j)~~ ——— ~~[illegible]~~

~~so subseq which for which~~

Now different ways in which we can argue

- could use T compact from $L^p \to H_0^1(\Omega)$ $p \geq 10$

  (u_j) bounded in L^p so $u_{j+1} = T(u_j)$ has conv. s-s. in H_0^1

- using Domin. convergence know monotone ptw conv. $u_j \to u$

  $u_j \to u$ in L^p $\forall p < \infty$

  so using only that $T: L^p \to H_0^1(\Omega)$ is continuous get that

  $u_{j+1} = T(u_j) \to T(u)$

  $u_j \to u$

  for full seq.

Variational approach:

Let $A(u)(v) = \int \nabla u \cdot \nabla v + \int u^5 \cdot v - \int v \, dx$

<u>Claim</u>: $A : H_0^1(\Omega) \to (H_0^1(\Omega))^*$ strictly monotone

<u>Proof</u>

$$\langle A(u) - A(v), u-v \rangle = \|\nabla(u-v)\|_{L^2}^2 + \underbrace{\int (u^5 - v^5)(u-v)}_{\geq 0}$$

have that $x \mapsto x^5$ strictly $\nearrow$ so

$$\geq \|\nabla(u-v)\|_{L^2}^2 \overset{\text{Poinc.}}{\geq} c_0 \|u-v\|_{H_0^1}^2.$$

<u>Also</u>: A maps bounded sets to bounded sets:

Let $R > 0$, $u \in H_0^1(\Omega)$ s.t. $\|u\|_{H^1} \leq R$

Then by Sobolev emb. $\exists\, C$ s.t.

$$\|u\|_{L^{10}} \leq C \cdot \|u\|_{H^1}$$

So $\forall v \in H_0^1$ s.t. $\|v\|_{H^1} = 1$ and hence $\|v\|_{L^2} \leq 1$

$$|A(u)(v)| \leq \|u^5\|_{L^2} \cdot \|v\|_{L^2} = \|u\|_{L^{10}}^5 \cdot \|v\|_{L^2} \leq R^5$$

Hence exist. & uniqueness follows from 1) & usual argument that sol. of ineq $\Rightarrow$ sol. of "=" as we are on a vect. space.

<u>Uniqueness</u>:

- from variat. approach ✓

or directly: If u, v are two sol. then using as testfn $\varphi = u - v$

$$\int (\nabla u - \nabla v)^2 + (u^5 - v^5)(u-v) = 0$$

so $u \equiv v$.

Using Schauder and sub/supersol. :

Know that

$$T : M \to M$$

$$M = \{ u \in H_0^1(\Omega) : \underline{u} \le u \le \bar{u} \} \subset H_0^1$$

since we can apply WMP to ineq. we obtain as above:

$$\begin{cases} (-\Delta+\lambda)(\underline{u}) \le (-\Delta+\lambda)\, u \le (-\Delta+\lambda)\, \bar{u} & \text{in } \Omega \\ \underline{u} \le u \le \bar{u} & \text{on } \partial\Omega \end{cases}$$

- $T(M)$ is precompact since
  $M \subset L^\infty(\Omega)$ bounded and

$$T: \; L^\infty(\Omega) \underset{\substack{\text{cont.} \\ \text{maps} \\ \text{bd sets to} \\ \text{bd sets}}}{\xrightarrow{u \mapsto f_\lambda(u)}} L^2(\Omega) \xrightarrow{\text{compact}} H_0^1(\Omega)$$

so Schauder vers II applies.