Problem Sheet 1

QUESTION 1. **Uniqueness of Solutions to ODEs.** Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$. Show that the initial value problem for $y\colon \mathbb{R} \to H$, given by

$$y'(t) = f(t, y(t)) \text{ for } t > 0, \quad y(0) = y_0,$$

has at most one continuously differentiable solution on the interval $[0,T]$, provided that $f\colon \mathbb{R} \times H \to H$ is continuous and satisfies for some $L > 0$

$$(f(t,y) - f(t,z), y - z) \leq L\|y - z\|^2 \text{ for all } y, z \in H. \tag{1}$$

[Hint: Use the product rule $\frac{d}{dt}(y(t), z(t)) = (y'(t), z(t)) + (z'(t), y(t))$ for functions $y, x\colon R \to H$ and Gronwall's Lemma.]

Give furthermore an example of a function f for which (1) is satisfied but for which the Lipschitz-condition of Picard's theorem does not hold.

QUESTION 2. **Null Lagrangian.**

(1) Give two examples of a Null-Lagrangian $L(\nabla u, u, x)$ (and explain in particular why the functions you propose are Null-Lagrangians.)

(2) Define for real $n \times n$ matrices $P \in \mathbb{R}^{n\times n}$ the map

$$L(P) = \mathrm{tr}(P^2) - (\mathrm{tr}(P))^2 .$$

where $\mathrm{tr}(P)$ denotes the trace of the matrix P. Show that L is a Null-Lagrangian.

QUESTION 3. **Euler-Lagrange Equations.**

(i) Let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be a domain. Derive the Euler-Lagrange equation for the functional

$$I(v) = \int_\Omega \frac{1}{p}|\nabla v|^p - \frac{1}{4}v^4 \, dx$$

where $v\colon \Omega \to \mathbb{R}$ and $|\nabla v| = \sqrt{(\partial_1 v)^2 + \ldots + (\partial_n v)^2}$ once by using the formula derived in the lecture and once by direct computation of $\frac{d}{dt}I(v + t\phi)$, $\phi \in C_c^\infty(\Omega)$.

(ii) Let $\Omega \subset\subset \mathbb{R}^3$ and $1 \leq p \leq 6$. Show that the functional

$$E(u) := \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^p}^2}$$

is well defined for all $u \in H_0^1(\Omega)$, $u \neq 0$ and satisfies $\inf\{E(u) : u \in H_0^1(\Omega)\} > 0$. Derive furthermore it's Euler-Lagrange equation.

Then consider

$$E_0(u) := \int |\nabla u|^2 dx$$

and explain what condition has to be satisfied for a function $u \in H_0^1(\Omega)$ which minimises E_0 in the set $M := \{v : \|v\|_{L^p} = 1\}$

QUESTION 4. **Counter-example to Brouwer's Fixed Point Theorem in an infinite dimensional space**. Consider the real Hilbert Space

$$l^2 = \left\{ (x_i)_{i\in\mathbb{N}} \text{ such that } \sum_{i=0}^\infty x_i^2 < \infty \right\} \text{ with the norm } \|x\|_{l^2} = \sqrt{\sum_{i=0}^\infty x_i^2}.$$

Let B be its closed unit ball.

- Consider the map

$$T\colon B \to B \text{ given by } T(x) = (\sqrt{1 - \|x\|_{l^2}^2}, x_0, x_1, x_2, \ldots).$$

Show that T is continuous and does not have a fixed point.

- Construct a continuous retraction from B to ∂B.

Question 5. **Equivalence between Retraction Principle and Brouwer's FPT.** Let B be the closed unit ball in $\mathbb{R}^n$. Using Brouwer's Fixed Point Theorem, show that there does not exist a retraction r from B to ∂B, i.e. a map $r : B \to \partial B$ such that r restricted to ∂B is the identity map.

Hint: by contradiction, consider the map $g(x) = -r(x)$.

Question 6. **Application of Brouwer's FPT.** Given a map $f \in C(\mathbb{R}^n : \mathbb{R}^n)$ such that $|f(x)| \leq a + b|x|$, with $a \geq 0$ and and $0 < b < 1$, show that f has a fixed point.