Problem Sheet 2

Question 1. **Leray-Schauder/Schaefer Theorem.**

- Prove the following result Let X be a Banach space and $T\colon X \to X$ be a compact map with the following property: there exists $R > 0$ such that the statement ($x = \tau Tx$ with $\tau \in [0,1)$) implies $\|x\|_X < R$. Then T has a fixed point x^* such that $\|x^*\|_X \leq R$.

  Hint: Consider the operators

$$T_n(x) := \begin{cases} Tx & \text{if } \|Tx\|_X \leq R + \frac{1}{n}, \\ \frac{R+1/n}{\|Tx\|_X} Tx & \text{else} \end{cases}$$

  on a suitable domain and prove that they are compact.
- Let $T : X \to X$ a compact map such that there exists $R > 0$ such that $\|Tx - x\|_X^2 \geq \|Tx\|_X^2 - \|x\|_X^2$ when $\|x\|_X \geq R$. Show that T admits a fixed point.

Question 2. **Leray's eigenvalue problem**. Let $K : [a, b] \times [a, b] \to (0, \infty)$ be a continuous and positive function and consider the integral operator $T : C^0([a, b]) \to C^0([a, b])$ defined by

$$(Tu)(x) = \int_a^b K(x,t)u(t)\,dt.$$

Prove that T has at least one non-negative eigenvalue λ whose eigenvector is a continuous non-negative function u, i.e. there exist $\lambda \geq 0$ and a non-negative u so that

$$\int_a^b K(x,t)u(t)\,dt = \lambda u(x).$$

Hint: consider, on an appropriate closed convex set M, the function

$$F(u) = \frac{1}{\int_a^b Tu(t)dt} \cdot Tu.$$

and apply one of the versions of Schauder's Fixed Point Theorem with the help of Arzéla-Ascoli Theorem. To find a suitable set M think about what property all functions $F(u)$ have in common.

Question 3. **Integral operators on $L^2(\Omega)$ vs. $C(\bar{\Omega})$** As always, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain.

- Let $a : \bar{\Omega} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous map, and let

$$A(u)(x) = \int_\Omega a(x, y, u(y))dy.$$

  show that $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined and compact. (Hint: use Arzela-Ascoli Theorem).
- Let $k \in L^2(\Omega \times \Omega)$ and define

$$(Ku)(x) = \int_\Omega k(x,y)u(y)dy.$$

  Show that $K : L^2(\Omega) \to L^2(\Omega)$ is well defined and compact. You can use for example that $C_0^\infty(\Omega \times \Omega)$ is dense in $L^2(\Omega \times \Omega)$, and therefore there is a sequence $k_m \in C_0^\infty(\Omega \times \Omega)$ such that $k_m \to k$ in $L^2(\Omega)$.
- Give an example of continuous a such that A (defined as above) is not well defined as an operator from $L^2(\Omega) \to L^2(\Omega)$.

QUESTION 4. **Continuous maps.** Let $g \in C(\mathbb{R} \times \mathbb{R}^n)$ be such that $g(z,p) \leq a + b|z|^\alpha + c|p|$, where a,b and c are non negative constants, and $2\alpha < 2^*$, where $2^* = 2n/(n-2)$ if $n \geq 3$, and $2^* = \infty$ if $n = 1, 2$. Then the map $u \mapsto g(u, \nabla u)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and maps bounded subsets of $H_0^1(\Omega)$ to bounded subsets of $L^2(\Omega)$.

Hint: rewrite $g(u, \nabla u) = \tilde{g}(u, \frac{\nabla u}{|\nabla u|^\nu})$ *for a suitable function* $\tilde{f}$ *and a suitable exponent* $0 < \nu < 1$ *and apply Lemma 2.7 from the lecture*