Problem Sheet 3

QUESTION 1. Let $f \in C^0(\mathbb{R}, \mathbb{R})$ be so that so that there exists some C such that $|f(u)| \leq C(1+|u|^{1/2})$. Modifying the proof of application 1 from the lecture, prove that

$$\Delta u = f(u) \text{ in } \Omega, \text{ with } u = 0 \text{ on } \partial\Omega,$$

has a weak solution.

QUESTION 2. **Proof of the weak Maximum Principle** Let $b \in L^\infty(\Omega, \mathbb{R}^n)$. Give a weak formulation of the condition

$$(\star) \qquad \Delta u + b \cdot \nabla u \leq 0$$

that is well defined for functions $u \in H^1(\Omega)$.

Then show that there exists a number $c_1 > 0$ so that if $u \in H^1(\Omega)$ satisfies the weak form of $(\star)$ for some b with $\|b\|_{L^\infty} \leq c_1$ and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$.

Hint: You may use that $u^- = -\min(u, 0) \in H^1_0(\Omega)$ with $\nabla u^- = -\nabla u \cdot \chi_{\{u<0\}}$ a.e.

QUESTION 3. **A non-linear PDE with a parameter**

Consider the PDE

$$-\Delta u = \exp\left(-\frac{\lambda}{u+1}\right) \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega.$$

Show that this problem can be formulated in an equivalent form so that it makes sense in $H^1_0(\Omega)$, i.e. making a modification to the right-hand-side that would not omit any solution, but would allow the equation to be well-posed. Prove that there exists at least one weak solution, for any λ and that this solution is unique if $\lambda < 0$.

QUESTION 4. **Sub and Super solutions.**

Given a smooth, bounded domain $\Omega \subset R^3$, we consider the following reaction-diffusion problem

$$-\Delta u + u(1-u) = -1 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega.$$

- Show that this problem makes sense, and in particular that it can be written (for any $\lambda > 0$) as a fixed point problem for $T := u \to (-\Delta + (\lambda+1))^{-1}(f(u) + \lambda u)$, where T is a continuous map on $H^1_0(\Omega)$.
- Find a sub-solution and a super-solution.
- Show that there exists a $\lambda > 0$ such that $u^2 - 1 + 2\lambda u$ is an increasing function of u when $u \geq \underline{u}$.
- Show that there exists at least one solution in $H^1_0(\Omega)$ by adapting the super/sub solution method given in the lecture notes.

QUESTION 5. **Frechet Derivative**

(a) For a smooth bounded domain Ω, consider the map $F : C^2(\bar{\Omega}) \to C(\bar{\Omega})$ given by

$$F(u) = \Delta u + f(u),$$

where $f \in C^1(R)$. Compute the directional derivatives of F, and show that F is Fréchet differentiable.

(b) Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 4$, be bounded and consider the function $F(u) := (-\Delta)^{-1}(u^2)$. Prove that F is a C^1 function from $H^1_0(\Omega)$ to $H^1_0(\Omega)$.