Lie groups and homogenous spaces : miniproject

The aim of this project is to study *homogeneous spaces*, manifolds with a *transitive* Lie group action.

The questions vary in difficulty and a high mark can be obtained without necessarily answering all of them. You are encouraged to include partial solutions if you cannot answer the question completely. You may also investigate other related questions that occur to you. If the meaning of a question is not clear you may ask for advice (email: dancer@maths.ox.ac.uk). You may refer to books and other sources but should give a reference to this in your solution.

1. Consider a manifold M on which a Lie group G acts transitively, such that the isotropy subgroup (stabiliser) of some $m \in M$ is equal to a closed subgroup K of G.

Explain why M can be identified G-equivariantly with the coset space G/K.

2. Show that there is a representation of K on the tangent space $T_{eK}(G/K)$ where eK denotes the identity coset.

We now assume K is *compact*.

3. Explain why the tangent space T_{eK} may be viewed as a complement (as a vector space) $\mathfrak{p}$ to $\mathfrak{k} = \mathrm{Lie}(K)$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as vector spaces, with K acting on $\mathfrak{p}$ as in Question 2.

Let $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \ldots \oplus \mathfrak{p}_r$ be the decomposition of $\mathfrak{p}$ into irreducible representations of K.

4. Show that the n-dimensional sphere S^n may be identified with the coset space $SO(n+1)/SO(n)$ and that in this case $\mathfrak{p}$ is itself an irreducible representation of $SO(n)$.

5. Recall that a Riemannian metric on M is a smooth choice of inner product on each tangent space T_mM to M. (So if X, Y are vector fields than $g(X, Y)$ is a smooth function on M).

Show that restriction to the tangent space at the identity coset defines a bijection between the set of G-invariant metrics on G/K and the set of K-invariant inner products on $\mathfrak{p}$.

6. Deduce that if $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \ldots \oplus \mathfrak{p}_r$ as above, and if moreover $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are inequivalent as K-representations for $i \neq j$, then the space of G-invariant metrics on G/K is r-dimensional.

In particular show that there is a unique $SO(n+1)$-invariant metric on S^n up to scale

7. Show that the $(2n-1)$-dimensional sphere may be viewed as the coset space $U(n)/U(n-1)$. Find the decomposition of $\mathfrak{p}$ into irreducibles in this case, and deduce the dimension of the space of $U(n)$-invariant metrics on S^{2n-1}.

8. Show that the bi-invariant metrics on G (ie those invariant under left and right translations) correspond to the inner products on $\mathfrak{g}$ invariant under the adjoint representation of G on $\mathfrak{g}$.

Deduce that every compact Lie group G admits a bi-invariant metric.

9. Recall (or find out about) the definition of the Levi-Civita connexion ∇ associated to a Riemannian metric. Show that for a bi-invariant metric we have, for left-invariant vector fields X, Y, the formula for this connexion

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

Show that the geodesics in this metric through the identity are the 1-parameter subgroups, and deduce that a bi-invariant metric is complete.