Lecture 5:

- Regular distributions
- The fundamental lemma of the calculus of variations
- Convergence in the sense of distributions

(pp. 28–31 in lecture notes)

Recall from previous lecture:

- A distribution of order at most m admits a unique extension to $C_c^m(\Omega)$ satisfying the boundedness property there. We use the same symbol for the extension.
- A distribution of order 0 extends uniquely to a Radon measure.

- A positive Radon measure is integration with respect to unique locally finite Borel measure (by Riesz-Markov)

- A general complex Radon measure can be uniquely identified with $\mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where μ_j are locally finite Borel measures

- A positive distribution is a locally finite Borel measure.

$\boxed{\text{DEF}}$ A distribution $u \in \mathcal{D}'(\Omega)$ is called a <u>regular distribution on Ω</u> if there exists $f \in L^1_{loc}(\Omega)$ so $u = T_f$, that is,

$$\langle u, \phi \rangle = \int_\Omega \phi f \, dx \ , \quad \phi \in \mathcal{D}(\Omega).$$

If $f, g \in L^1_{loc}(\Omega)$ and $T_f = T_g$ does it follow that $f = g$ a.e.?

— Yes, by

<u>The fundamental lemma of the calculus of variations</u> (or <u>Du Bois-Reymond lemma</u>).

If $f \in L^1_{loc}(\Omega)$ and $\int_\Omega f\phi\,dx = 0$ for all $\phi \in \mathcal{D}(\Omega)$, then $f = 0$ a.e.

$\boxed{\text{Pf}}$ Take non-empty $\mathcal{O} \Subset \Omega$ (this notation stands for: $\mathcal{O}$ open, $\overline{\mathcal{O}}$ compact and $\overline{\mathcal{O}} \subset \Omega$).

Put $g = f\mathbb{1}_{\mathcal{O}} := \begin{cases} f & \text{in } \mathcal{O} \\ 0 & \text{in } \mathbb{R}^n \setminus \mathcal{O} \end{cases}$.

Since $\overline{\mathcal{O}}$ is compact, $g \in L^1(\mathbb{R}^n)$.

If $(\rho_\varepsilon)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}^n$, then

$$\|\rho_\varepsilon * g - g\|_1 \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0$$

(by Prop. 2.7 in lecture notes).

Now fix $x \in \mathcal{O}$. Then

$$(\rho_\varepsilon * g)(x) = \int_{\mathcal{O}} \rho_\varepsilon(x-y) f(y)\, dy$$

and if $\varepsilon < \operatorname{dist}(x, \partial\mathcal{O})$, then $B_\varepsilon(x) \subset \mathcal{O}$. Therefore

$$\phi^x(y) := \rho_\varepsilon(x-y), \quad y \in \Omega,$$

is C^∞ and $\operatorname{supp} \phi^x = \overline{B_\varepsilon(x)} \subset \Omega$. In particular, $\phi^x \in \mathcal{D}(\Omega)$ and $\operatorname{supp} \phi^x \subset \mathcal{O}$, so

$$(\rho_\varepsilon * g)(x) = \int_\Omega \phi^x f\, dy = 0.$$

Thus $(\rho_\varepsilon * g)(x) \to 0$ as $\varepsilon \to 0$ pointwise in $x \in \mathcal{O}$, and so using Fatou's Lemma

$$\int_{\mathcal{O}} |g|\,dx \leqq \liminf_{\varepsilon \to 0} \int_{\mathcal{O}} |\rho_\varepsilon * g - g|\,dx$$

$$\leqq \liminf_{\varepsilon \to 0} \|\rho_\varepsilon * g - g\|_1 = 0.$$

It follows that $g = 0$ a.e. on $\mathcal{O}$, that is, $f = 0$ a.e. on $\mathcal{O}$, and since $\mathcal{O} \Subset \Omega$ was arbitrary we're done. $\square$

Lemma If μ and ν are locally finite Borel measures on Ω and $T_\mu = T_\nu$, then $\mu = \nu$.

$\boxed{\text{Pf}}$ It suffices to show that $\mu(K) = \nu(K)$ for all compact $K \subset \Omega$. (Why? — Exercise)

Put $\phi_\varepsilon = \rho_\varepsilon * \mathbb{1}_K$. If $\varepsilon < \text{dist}(K, \partial\Omega)$ then $\phi_\varepsilon \in \mathcal{D}(\Omega)$ and so

$$\int_\Omega \phi_\varepsilon \, d\mu = T_\mu(\phi_\varepsilon) = T_\nu(\phi_\varepsilon) = \int_\Omega \phi_\varepsilon \, d\nu.$$

Note $0 \leq \phi_\varepsilon \leq \mathbb{1}_{\overline{B_{\varepsilon_0}(K)}}$, $\varepsilon < \varepsilon_0$, $\varepsilon_0 = \frac{1}{2}\text{dist}(K, \partial\Omega)$, and

$$\mu(\overline{B_{\varepsilon_0}(K)}), \ \nu(\overline{B_{\varepsilon_0}(K)}) < \infty,$$

so taking $\varepsilon \searrow 0$ we deduce from Lebesgue's dominated con= vergence theorem

$$\mu(K) = \nu(K). \quad \square$$

This settles the uniqueness in the last result from lecture 4.

Notation

When $f \in L^1_{loc}(\Omega)$ we identify the distribution T_f with f and write

$$\boxed{T_f = f}$$

When μ is a locally finite Borel measure on Ω we identify the distribution T_μ with μ and write

$$\boxed{T_\mu = \mu}$$

What is meant will then be clear from context.

$\boxed{\text{DEF}}$ **Convergence in the sense of distributions on Ω**

Let (u_j) be a sequence in $\mathcal{D}'(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Then

$$u_j \to u \quad \text{in} \quad \mathcal{D}'(\Omega)$$

if

$$\langle u_j, \phi \rangle \to \langle u, \phi \rangle$$

for all $\phi \in \mathcal{D}(\Omega)$.

Remarks

As with 'convergence in the sense of test functions on Ω' there exists a topology τ' on $\mathcal{D}'(\Omega)$ so $(\mathcal{D}'(\Omega), \tau')$ is a topological vector space and $u_j \to u$ in $\mathcal{D}'(\Omega)$ precisely if $u_j \to u$ in $(\mathcal{D}'(\Omega), \tau')$.

In fact, $(\mathcal{D}'(\Omega), \tau')$ is the topologically dual space to $(\mathcal{D}(\Omega), \tau)$.

We won't pursue this here.

Whereas 'convergence in $\mathcal{D}(\Omega)$' was a very strong condition 'convergence in $\mathcal{D}'(\Omega)$' is a very weak condition.

$\boxed{\text{EX}}$ Let $f_j, f \in L^p(\Omega)$, $p \in [1, \infty]$.

If $f_j \to f$ in $L^p(\Omega)$, so

$$\|f_j - f\|_p \longrightarrow 0,$$

then also $f_j \longrightarrow f$ in $\mathcal{D}'(\Omega)$.

Recall notation: $f_j = T_{f_j}$ ....

Indeed, if $\phi \in \mathcal{D}(\Omega)$, then

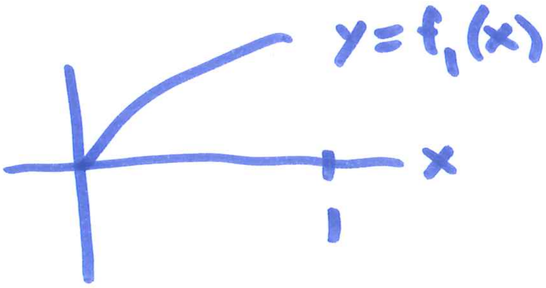
$$|\langle f_j, \phi\rangle - \langle f, \phi\rangle| \leq \int_\Omega |f_j - f| \cdot |\phi| \, dx$$

$$\leq \|f_j - f\|_p \|\phi\|_q \quad \text{by Hölder} \quad \frac{1}{p} + \frac{1}{q} = 1 .$$

The converse is false.

$\boxed{\text{EX}}$ $f_j(x) = \sin(jx)$, $x \in (0,1)$.

Clearly $\|f_j\|_p \not\to 0$ for $p \in [1,\infty]$.



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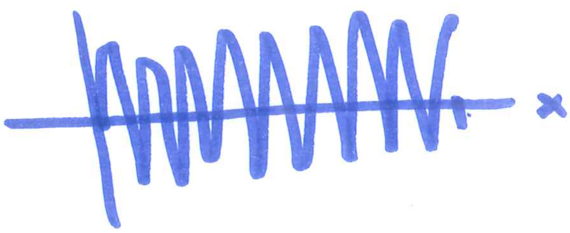

j large

Let $\phi \in \mathcal{D}(0,1)$. Then

$$\langle f_j, \phi\rangle = \int_0^1 \sin(jx)\phi(x)\,dx \overset{\text{parts}}{=}$$

$$\left[-\frac{\cos(jx)}{j}\phi(x)\right]_0^1 + \int_0^1 \frac{\cos(jx)}{j}\phi'(x)\,dx \to 0$$

hence $f_j \to 0$ in $\mathcal{D}'(0,1)$.

$\boxed{\text{EX}}$ $g_j(x) = g(jx)$, $x \in (0,1)$,

where for a $T>0$, $g : \mathbb{R} \to \mathbb{C}$ is T-periodic and

$$g(x) = \begin{cases} -117 & \text{in } (0, \frac{T}{2}] \\ +117 & \text{in } (\frac{T}{2}, T] \end{cases}$$

Note $|g_j(x)| = 117$ for all x, but integration by parts shows

$g_j \to 0$ in $\mathcal{D}'(0,1)$.

More precisely: if $\phi \in \mathcal{D}(0,1)$ then since g is piecewise continuous we get with

$$G(x) = \int_0^x g(t)\,dt, \quad x \in (0,T],$$

that $G(T)=0$, so the T-periodic extension of G is piecewise C^1 and $G'(x)=g(x)$ for $x \notin \frac{T}{2}\mathbb{Z}$.

Therefore we may integrate by parts to find

$$\langle g_j, \phi \rangle = \int_0^1 g(jx)\phi(x)\,dx =$$

$$\left[\frac{G(jx)}{j}\phi(x) \right]_0^1 - \int_0^1 \frac{G(jx)}{j}\phi'(x)\,dx$$

$$\to 0.$$

$\boxed{\text{EX}}$ $h_j(x) = j\,g(jx)$, $x \in (0,1)$,

with g as above.

For $\phi \in \mathcal{D}(0,1)$ we get as in previous ex:

$$\langle h_j, \phi \rangle = \left[G(jx)\phi(x) \right]_0^1 - \int_0^1 G(jx)\phi'(x)\,dx$$

hence

$$\langle h_j, \phi \rangle = -\int_0^1 G(jx)\,\phi'(x)\,dx .$$

Put $\langle G \rangle = \frac{1}{T}\int_0^T G(t)\,dt = 117\,\frac{T^2}{4}$

and $H(x) = \int_0^x \big(G(t) - \langle G \rangle\big)\,dt, \; x \in (0,T]$.

Then $H(0^+) = H(T) = 0$ so the T-periodic extension of H to $\mathbb{R}$ is piecewise C^1 and integration by parts gives

$$\langle h_j, \phi \rangle = -\int_0^1 \big(G(jx) - \langle G \rangle\big)\,\phi'(x)\,dx$$

$$= -\Big[\tfrac{1}{j} H(jx)\,\phi'(x)\Big]_0^1 + \int_0^1 \tfrac{1}{j} H(jx)\,\phi''(x)\,dx$$

$$\longrightarrow 0 .$$

Thus $h_j \to 0$ in $\mathcal{D}'(0,1)$.

$\boxed{\text{EX}}$ $(\rho_\varepsilon)_{\varepsilon>0}$ the standard mollifier on $\mathbb{R}^n$. Then

$$\rho_\varepsilon \to \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \quad \text{as } \varepsilon \searrow 0$$

If $\phi \in \mathcal{D}(\mathbb{R}^n)$, then

$$\langle \rho_\varepsilon, \phi \rangle = \int_{\mathbb{R}^n} \rho_\varepsilon(x)\phi(x)\,dx \overset{\rho_\varepsilon \text{ even}}{=}$$

$$\int_{\mathbb{R}^n} \rho_\varepsilon(0-x)\phi(x)\,dx = (\rho_\varepsilon * \phi)(0)$$

$$\to \phi(0) = \langle \delta_0, \phi \rangle .$$

We can easily generalize this.

Let $v \in C_c(\mathbb{R}^n)$ and put

$$v_\varepsilon(x) = \varepsilon^{-n} v(\varepsilon^{-1}x), \quad x \in \mathbb{R}^n,$$

be the L^1-dilation of v.

For $\phi \in \mathcal{D}(\mathbb{R}^n)$:

$$\langle v_\varepsilon, \phi \rangle = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} v\left(\frac{x}{\varepsilon}\right) \phi(x)\, dx$$

$y = \frac{x}{\varepsilon}$

so $\underline{\underline{\varepsilon^n}} dy = dx$

$$\int_{\mathbb{R}^n} v(y)\, \phi(\varepsilon y)\, dy$$

$$\xrightarrow[\varepsilon \to 0]{} \int_{\mathbb{R}^n} v(y)\, dy\ \phi(0) \qquad \text{(why?)}$$

hence $v_\varepsilon \to \int_{\mathbb{R}^n} v(y)\, dy\ \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.