B4.3

Distribution Theory MT20

Lecture 6:
- Adjoint identity scheme
- Differentiation in the sense of distributions
- Convolution with test function

(pp. 31–36 in lecture notes)

Given an operation on test functions, say differentiation, we would like to extend it to distributions.

Suppose T is an operation on test functions that we would like to extend to distributions:

$$\emptyset \neq \Omega \subseteq \mathbb{R}^n \text{ open subset}$$

$$T : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega) \text{ linear map}$$

Assume there exists a linear and $\mathcal{D}$-continuous ① map

$$S : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$$

such that

$$\circledast \quad \int_\Omega T(\phi)\psi \, dx = \int_\Omega \phi S(\psi) \, dx$$

for all $\phi, \psi \in \mathcal{D}(\Omega)$.

We refer to $\circledast$ as an <u>adjoint identity</u>

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① if $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$, then $S(\phi_j) \to S(\phi)$ in $\mathcal{D}(\Omega)$.

Define for $u \in \mathcal{D}'(\Omega)$

$$\langle \overline{T}(u), \phi \rangle := \langle u, S(\phi) \rangle, \ \phi \in \mathcal{D}(\Omega).$$

Then $\overline{T}(u) : \mathcal{D}(\Omega) \to \mathbb{C}$ is linear and $\mathcal{D}$-continuous:

$$\langle \overline{T}(u), \phi + \lambda \psi \rangle = \langle \overline{T}(u), \phi \rangle + \lambda \langle \overline{T}(u), \psi \rangle$$

since u, S are linear,

$$\langle \overline{T}(u), \phi_j \rangle \to 0 \quad \text{if } \phi_j \to 0 \text{ in } \mathcal{D}(\Omega)$$

since S, u are $\mathcal{D}$-continuous.

Thus $\overline{T}(u) \in \mathcal{D}'(\Omega)$, and therefore

$$\overline{T} : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$$

By inspection we see $\overline{T}$ is linear and if $u_j \to u$ in $\mathcal{D}'(\Omega)$, then

$$\langle \overline{T}(u_j), \phi \rangle = \langle u_j, S(\phi) \rangle \to \langle u, S(\phi) \rangle$$
$$= \langle \overline{T}(u), \phi \rangle \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Hence $\overline{T}: \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is linear and $\mathcal{D}'$-continuous. The adjoint identity ⊛ ensures that $\overline{T}$ is an extension of T:

Let $\psi \in \mathcal{D}(\Omega)$. We may consider ψ to be a distribution $(= T_\psi)$ and get from ⊛ for $\phi \in \mathcal{D}(\Omega)$

$$\langle \overline{T}(T_\psi), \phi \rangle = \langle T_\psi, S(\phi) \rangle =$$

$$\int_\Omega \psi\, S(\phi)\, dx = \int_\Omega T(\psi)\, \phi\, dx$$

hence (using $\psi = T_\psi$) $\overline{T}(\psi) = T(\psi)$.

$\boxed{\text{Terminology}}$ When we extend an operation on test functions to distributions by use of the above procedure we say we do it by the

$\boxed{\text{EX}}$ Differentiation

Consider $D = \frac{d}{dx}$ on $\mathcal{D}(\mathbb{R})$.

Note $D : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is linear (and $\mathcal{D}$-continuous).

Can we extend it to distributions?

Need an adjoint identity!

For $\phi, \psi \in \mathcal{D}(\mathbb{R})$ we calculate by integration by parts:

$$\int_{-\infty}^{\infty} D(\phi)\psi \, dx = \int_{-\infty}^{\infty} \phi' \psi \, dx =$$

$$\Big[\phi\psi\Big]_{x\to-\infty}^{x\to\infty} - \int_{-\infty}^{\infty} \phi\psi' \, dx = -\int_{-\infty}^{\infty} \phi\psi' \, dx$$

Adjoint identity with $T = D$, $S = -D$.

We note that $S = -D : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is linear and $\mathcal{D}$-continuous as required and so if $u \in \mathcal{D}'(\mathbb{R})$,

$$\langle \bar{D}u, \phi \rangle := \langle u, -\phi' \rangle, \quad \phi \in \mathcal{D}(\mathbb{R}),$$

is an extension of $D = \frac{d}{dx}$ to distributions. The adjoint identity scheme ensures that

$$\bar{D}T_\phi = T_{D\phi} \qquad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

What happens when $u \in C'(\mathbb{R})$?

For $\phi \in \mathcal{D}(\mathbb{R})$:

$$\langle \bar{D}T_u, \phi \rangle = \langle T_u, -\phi' \rangle =$$

$$\int_{-\infty}^{\infty} u(-\phi')\,dx \overset{\text{parts}}{=} \Big[u(-\phi) \Big]_{x \to -\infty}^{x \to \infty} + \int_{-\infty}^{\infty} u'\phi\,dx$$

$$= \int_{-\infty}^{\infty} u'\phi\,dx = \langle T_{u'}, \phi \rangle$$

$$\therefore \bar{D}T_u = T_{u'}$$

We get in other words nothing new: when u is C^1 the distributional derivative is just the usual derivative!

Because of this we write $Du = u'$ also when u is a distribution. Thus distributional derivatives are denoted like classical derivatives, what is meant is clear from context, or will be emphasized.

EX Multiplication with C^∞ functions

Let $f \in C^\infty(\mathbb{R})$.
If $T(\phi) := f\phi$, then $T : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is linear (and $\mathcal{D}$-continuous, by generalized Leibniz rule).

We clearly have an adjoint identity with $S = T$:

$$\int_{-\infty}^{\infty} (f\phi)\psi \, dx = \int_{-\infty}^{\infty} \phi (f\psi) \, dx \quad \forall \phi, \psi \in \mathcal{D}(\mathbb{R})$$

We can then define for $u \in \mathcal{D}'(\mathbb{R})$

$$\langle fu, \phi \rangle := \langle u, f\phi \rangle, \quad \phi \in \mathcal{D}(\mathbb{R}).$$

The adjoint identity scheme ensures consistency for $u \in \mathcal{D}(\mathbb{R})$, but in this case it's clear we have consistency for all regular distri= butions: if $u \in L^1_{loc}(\mathbb{R})$, then

$$f T_u = T_{fu}.$$

Note also that we also can define uf and that we trivially have

$$uf = fu \qquad \text{for } u \in \mathcal{D}'(\mathbb{R}), f \in C^\infty(\mathbb{R})$$

$\boxed{\text{EX}}$ Many other operations on test functions can be extended to distributions:

<u>Composition with C^∞ diffeomorphism</u>

$\Phi : \mathbb{R} \to \mathbb{R}$ C^∞ diffeomorphism

(that is, $\Phi \in C^\infty$ bijective and $\Phi' \neq 0$)

Note $\phi \circ \Phi \in \mathcal{D}(\mathbb{R})$ if $\phi \in \mathcal{D}(\mathbb{R})$

and
$$T : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R}),$$
$$T(\phi) := \phi \circ \Phi$$
is linear. Extend it to distributions?

Need adjoint identity: for $\phi, \psi \in \mathcal{D}(\mathbb{R})$

calculate using substitution formula

$$\int_{-\infty}^{\infty} \phi \circ \Phi \, \psi \, dx = \left| \begin{array}{l} y = \Phi(x) \text{ so} \\ x = \Phi^{-1}(y) \text{ and} \\ dx = \dfrac{1}{\Phi'(\Phi^{-1}(y))} dy \end{array} \right.$$

$$\int_{\Phi(-\infty)}^{\Phi(\infty)} \phi \, \psi \circ \Phi^{-1} \frac{dy}{\Phi'(\Phi^{-1})} =$$

$$\int_{-\infty}^{\infty} \phi \frac{\psi \circ \Phi^{-1}}{|\Phi' \circ \Phi^{-1}|} dy .$$

Adjoint identity with

$$S(\psi) = \frac{\psi \circ \Phi^{-1}}{|\Phi' \circ \Phi^{-1}|} .$$

Note $S : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ linear and $\mathcal{D}$-continuous (one must use both chain and Leibniz rules for verification of this).

We write $\Phi_* u$ for $u \circ \Phi$ when $u \in \mathcal{D}'(\mathbb{R})$. Thus:

$$\langle \Phi_* u, \phi \rangle := \left\langle u, \frac{\phi \circ \Phi^{-1}}{|\Phi' \circ \Phi^{-1}|} \right\rangle, \phi \in \mathcal{D}(\mathbb{R}).$$

We have consistency for regular distributions: if $u \in L^1_{loc}(\mathbb{R})$, then

$$\Phi_* T_u = T_{u \circ \Phi} .$$

<u>Important special cases:</u>

<u>Translation</u> $\Phi(x) := x + h$ $(h \in \mathbb{R})$

Translation by h is denoted τ_h

$$\langle \tau_h u, \phi \rangle := \langle u, \tau_{-h} \phi \rangle, \phi \in \mathcal{D}(\mathbb{R}).$$

<u>Dilation</u> $\Phi(x) := rx$ $(r > 0)$

Dilation by r is denoted d_r

$$\langle d_r u, \phi \rangle := \langle u, \tfrac{1}{r} d_{\frac{1}{r}} \phi \rangle, \ \phi \in \mathcal{D}(\mathbb{R})$$

Note that here $\frac{1}{r} d_{\frac{1}{r}}$ is L^1-dilation with r:

$$\left(\frac{1}{r} d_{\frac{1}{r}} \phi\right)(x) = \frac{1}{r}\phi\left(\frac{x}{r}\right)$$

Check that $\langle \frac{1}{r} d_{\frac{1}{r}} u, \phi \rangle = \langle u, d_r \phi \rangle$ holds for all $u \in \mathcal{D}'(\mathbb{R}), \phi \in \mathcal{D}(\mathbb{R}), r > 0$.

<u>Reflection in origin</u> $\Phi(x) := -x$

Reflection in origin is denoted $\widetilde{(\cdot)}$

$$\langle \tilde{u}, \phi \rangle := \langle u, \tilde{\phi} \rangle, \quad \phi \in \mathcal{D}(\mathbb{R})$$

(so $\tilde{\phi}(x) := \phi(-x), \quad x \in \mathbb{R}$)

$\boxed{\text{EX}}$ <u>Convolution with test function</u>

Fix $\theta \in \mathcal{D}(\mathbb{R})$.

Put $T\phi := \theta * \phi, \quad \phi \in \mathcal{D}(\mathbb{R})$.

Then $T : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ linear (and $\mathcal{D}$-continuous)

Adjoint identity obtained by use of Fubini: if $\phi, \psi \in \mathcal{D}(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} \theta * \phi \, \psi \, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta(x-y)\phi(y)\psi(x)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta(x-y)\phi(y)\psi(x)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \phi \, \tilde{\theta} * \psi \, dy$$

Note $S(\psi) := \tilde{\theta} * \psi$, $S : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is linear and $\mathcal{D}$-continuous.

By adjoint identity scheme we may define for $u \in \mathcal{D}'(\mathbb{R})$:

$$\langle \theta * u, \phi \rangle := \langle u, \tilde{\theta} * \phi \rangle, \quad \phi \in \mathcal{D}(\mathbb{R}).$$

We have consistency when u is a regular distribution: $u \in L^1_{loc}(\mathbb{R})$

$\theta * T_u = T_{\theta * u}$ holds.

Note also that we could similarly define $u * \theta$ for $u \in \mathcal{D}'(\mathbb{R}), \theta \in \mathcal{D}(\mathbb{R})$ but $u * \theta = \theta * u$ so this is nothing new (exercise: check it)

We highlight the definitions of differentiation, multiplication by C^∞ function and convolution with test function in the general n-dimensional case:

$\boxed{\text{DEF}}$ Let $u \in \mathcal{D}'(\Omega)$, $1 \leq j \leq n$.

Then $\langle \bar{\partial}_j u, \phi \rangle := -\langle u, \partial_j \phi \rangle, \phi \in \mathcal{D}(\Omega)$.

As in 1-dimensional case we have consistency for $u \in C^1(\Omega)$:

$$\bar{\partial}_j T_u = T_{\partial_j u}$$

We therefore use the same notation for distributional partial derivatives as for classical partial derivatives:

$$\bar{\partial}_j u = \partial_j u = \frac{\partial u}{\partial x_j} \quad \text{etc}$$

What is intended will be clear from context or must be mentioned.

Note that $\partial_j \partial_k \phi = \partial_k \partial_j \phi$ for $\phi \in \mathcal{D}(\Omega)$, so also $\partial_j \partial_k u = \partial_k \partial_j u$ for $u \in \mathcal{D}'(\Omega)$. We can therefore also use multi-index notation for distri= butional partial derivatives:

for $u \in \mathcal{D}'(\Omega)$, $\alpha \in \mathbb{N}_0^n$

$$\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle, \phi \in \mathcal{D}(\Omega).$$

$\boxed{\text{DEF}}$ Let $u \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega)$.

Then $\langle fu, \phi \rangle := \langle u, f\phi \rangle$, $\phi \in \mathcal{D}(\Omega)$.

As in 1-dimensional case we have consistency for regular distributions:

$$f\, T_u = T_{fu} \quad \text{if } u \in L^1_{loc}(\Omega).$$

We also have $fu = uf$.

$\boxed{\text{Leibniz rule}}$ $u \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega)$.

$$\partial_j (fu) = (\partial_j f) u + f \partial_j u$$

and for $\alpha \in \mathbb{N}_0^n$

$$\partial^\alpha (fu) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f\, \partial^{\alpha - \beta} u$$

Pf by use of defs and Leibniz for test functions.

$\boxed{\text{DEF}}$ Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $\theta \in \mathcal{D}(\mathbb{R}^n)$.

Then $\langle \theta * u, \phi \rangle := \langle u, \tilde{\theta} * \phi \rangle$, $\phi \in \mathcal{D}(\mathbb{R}^n)$

As in 1-dimensional case we have consistency for regular distributions:

$$\theta * T_u = T_{\theta * u} \quad \text{if } u \in L^1_{loc}(\mathbb{R}^n)$$

Note: when $u \in L^1_{loc}(\mathbb{R}^n)$, $\theta \in \mathcal{D}(\mathbb{R}^n)$ then $\theta * u$ is well-defined

$$(\theta * u)(x) = \int_{\mathbb{R}^n} \theta(x-y)u(y)\,dy$$

$$= \int_{x - \operatorname{supp}\theta} \theta(x-y)u(y)\,dy$$

a compact set!

Remark we also have $\theta * u = u * \theta$ for $u \in \mathcal{D}'(\mathbb{R}^n)$, $\theta \in \mathcal{D}(\mathbb{R}^n)$.