# Lecture 10: Calculus of distributions

- Positive distributions revisited
- Characterization of monotone functions
- Differential inequalities, sub- and super-solutions

(pp. ~~48~~–51 in lecture notes)

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Recall that $u \in \mathcal{D}'(\Omega)$ is positive if $\langle u, \phi \rangle \geqq 0$ holds when $\phi \in \mathcal{D}(\Omega)$ and $\phi(x) \geqq 0$ for all $x \in \Omega$.

We have seen that a positive distribution is a measure.

More precisely, there exists a unique locally finite Borel measure $\mu$ on $\Omega$ so

$$\langle u, \phi \rangle = \int_\Omega \phi \, d\mu \,, \quad \phi \in \mathcal{D}(\Omega).$$

This identity extends to hold for $\phi \in C_c(\Omega)$ and we also write $u$ for this extension of $u$. (We proved that $u$ extends to a unique Radon measure on $\Omega$ and then we used Riesz-Markov.)

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<u>Observation</u> related to the fundamental lemma of the calculus of variations:

If $u \in L^1_{loc}(\Omega)$, then $u$ is positive as a distribution iff $u(x) \geqq 0$ a.e.

'$\Leftarrow$' is clear.

'$\Rightarrow$' Assume $\int_\Omega u\phi \, dx \geqq 0$ for all $\phi \in \mathcal{D}(\Omega)$ with $\phi \geqq 0$.

By contradiction: suppose

$$\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}) > 0$$

Since $\{u<0\} = \bigcup_{k=1}^{\infty} \{u < -\frac{1}{k}\}$ we find $k \in \mathbb{N}$ so

$$\mathcal{L}^n(\{u < -\tfrac{1}{k}\}) > \tfrac{1}{2} \mathcal{L}^n(\{u<0\} \cap B_k(0))$$

$$> 0.$$

Next by inner regularity of $\mathcal{L}^n$ we find a compact subset

$$K \subset \{u < -\tfrac{1}{k}\}$$

so $\mathcal{L}^n(K) > \frac{1}{2}\mathcal{L}^n(\{u<0\} \cap B_k(0)) > 0.$

For the standard mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ consider $\rho_\varepsilon * \mathbb{1}_K$ :

$$0 \leq \rho_\varepsilon * \mathbb{1}_K \in \mathcal{D}(\Omega) \text{ if } \varepsilon < \operatorname{dist}(K, \partial\Omega)$$

hence for such $\varepsilon$,

$$0 \leq \langle u, \rho_\varepsilon * \mathbb{1}_K \rangle = \int_\Omega u \, \rho_\varepsilon * \mathbb{1}_K \, dx .$$

Fix $d = \operatorname{dist}(K, \partial\Omega)$ and note that if $\varepsilon \in (0, \frac{d}{2})$,

$$|u \, \rho_\varepsilon * \mathbb{1}_K| \leq |u| \, \mathbb{1}_{\overline{B_{d/2}(K)}} \in L^1(\Omega)$$

compact $\subset \Omega$

and $u \, \rho_\varepsilon * \mathbb{1}_K \longrightarrow u \, \mathbb{1}_K$ pointwise a.e. as $\varepsilon \searrow 0$ through suitable subsequence

(since from previous lecture

$$\|\rho_\varepsilon * \mathbb{1}_K - \mathbb{1}_K\|_1 \longrightarrow 0 \text{ as } \varepsilon \searrow 0).$$

By Lebesgue's dominated convergence theorem

$$\int_\Omega u \, \rho_\varepsilon * \mathbb{1}_K \, dx \longrightarrow \int_\Omega u \, \mathbb{1}_K \, dx$$

as $\varepsilon \to 0$ through suitable subsequence.

Consequently,

$$0 \leqq \int_K u\,dx \underset{\substack{\uparrow \\ u<-\frac{1}{k} \text{ on } K}}{\leqq} -\frac{1}{k}\mathcal{L}^n(K) \quad \lightning$$

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Recall $f:(a,b)\to\mathbb{R}$ is monotone if it is either increasing or decreasing.

A monotone function is Borel measurable and locally bounded:

- for each $y\in\mathbb{R}$, $f^{-1}(y,\infty)$ is an interval (in the wide sense) hence in particular a Borel set

- for $[c,d]\subset(a,b)$ we have

$$\sup_{[c,d]}|f| \leqq \max\{|f(c)|,|f(d)|\}$$

Therefore a monotone function may be considered as a regular distribution.

We recall that the mean value theorem implies that a differentiable function $f:(a,b)\to\mathbb{R}$ is increasing iff $f'(x)\geq 0$ for all $x\in(a,b)$.

In general an increasing function can be discontinuous. However it is not entirely bad. Assume $f:(a,b)\to\mathbb{R}$ is increasing. Then we recall:

- $D=\{x\in(a,b): f \text{ not cont. at } x\}$ is at most countable,

- if $x_0\in D$, then

$$f(x_0-)=\lim_{x\to x_0-} f(x), \quad f(x_0+)=\lim_{x\to x_0+} f(x)$$

both exist in $\mathbb{R}$, and

$$f(x_0-)\leq f(x_0)\leq f(x_0+)$$

$\therefore$ all discontinuities of $f$ are jump discontinuities

In particular, by changing the values of $f$ at each $x \in D$ to $\lim_{y \to x+} f(y)$ we obtain a right-continuous (increasing) function. If therefore a regular distribution is given by an increasing function, then we can always find a right-continuous (increasing) represen= tative. It's not difficult to see that it's uniquely determined at all $x$:

Indeed, if $f, g : (a,b) \to \mathbb{R}$ are right-continuous and $f(x_0) \neq g(x_0)$ for some $x_0 \in (a,b)$, then also $\mathcal{L}^1(\{x \in (a,b) : f(x) \neq g(x)\}) > 0$.

Clearly a distribution $u \in \mathcal{D}'(a,b)$ that is defined by a monotone function also admits non-monotone represen= tatives. But by the above it has a unique right-continuous (monotone) representative.

$\boxed{\text{TH}}$ Let $u \in \mathcal{D}'(a,b)$. Then $u$ is defined by an increasing function iff $u' \geqq 0$ in $\mathcal{D}'(a,b)$.

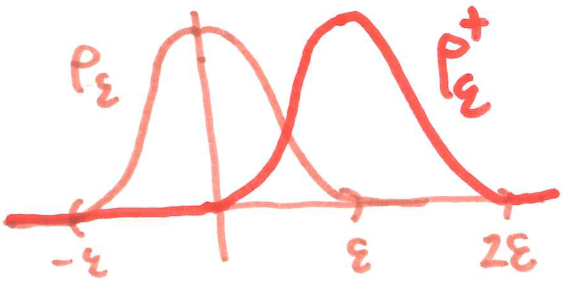
$\boxed{\text{Pf.}}$ We assume $(a,b) = \mathbb{R}$ and use mollification. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$.

Step 1. Assume $u \in C^\infty(\mathbb{R})$

Clearly, $u$ increasing iff $u'(x) \geqq 0$ for all $x \in \mathbb{R}$ iff $u' \geqq 0$ in $\mathcal{D}'(\mathbb{R})$.

Preample to step 2: Assume $u \in \overset{\infty}{C}(\mathbb{R})$ is increasing.

Define $\rho_\varepsilon^+(x) := \rho_\varepsilon(x-\varepsilon)$, $x \in \mathbb{R}$




$\rho_\varepsilon^+ \geqq 0$, $\int_{-\infty}^{\infty} \rho_\varepsilon^+ \, dx = 1$

and $\operatorname{supp}(\rho_\varepsilon^+) = [0, 2\varepsilon]$

$(\rho_\varepsilon^+)_{\varepsilon>0}$ is an approximate identity

and for $\phi \in \mathcal{D}(\mathbb{R})$ we have

$$\rho_\varepsilon^+ * \phi \longrightarrow \phi \quad \text{in } \mathcal{D}(\mathbb{R}) \text{ as } \varepsilon \searrow 0$$

and $(\rho_\varepsilon^+ * u)(x) \to u(x)$ locally uniformly in $x \in \mathbb{R}$ as $\varepsilon \searrow 0$.

Note in particular

$$(\rho_\varepsilon^+ * u)(x) = \int_0^2 u(x - \varepsilon y)\rho_1^+(y)\,dy$$

is an increasing function of $x \in \mathbb{R}$ and a decreasing function of $\varepsilon > 0$.

Step 2. General case $u \in \mathcal{D}'(\mathbb{R})$

Assume $u' \geqq 0$ in $\mathcal{D}'(\mathbb{R})$.

Then $\rho_\varepsilon * u \in C^\infty(\mathbb{R})$,

$$(\rho_\varepsilon * u)' = \rho_\varepsilon * u' \geqq 0$$

so $\rho_\varepsilon * u$ is an increasing function.

Let $\bar{\varepsilon} > 0$ and consider

$$(\rho_\varepsilon * u) * \rho^+_{\bar{\varepsilon}}.$$

For fixed $\varepsilon > 0$ we have that

$$[(\rho_\varepsilon * u) * \rho^+_{\bar{\varepsilon}}](x)$$

is an increasing function in $x \in \mathbb{R}$ and a decreasing function in $\bar{\varepsilon} > 0$.

Because

$$[(\rho_\varepsilon * u) * \rho^+_{\bar{\varepsilon}}](x) \xrightarrow[\varepsilon \searrow 0]{} (u * \rho^+_{\bar{\varepsilon}})(x)$$

pointwise in $x \in \mathbb{R}$, $\bar{\varepsilon} > 0$ the limit is increasing in $x \in \mathbb{R}$ and decreasing in $\bar{\varepsilon} > 0$.

For fixed $x \in \mathbb{R}$, $(u * \rho^+_{\bar{\varepsilon}})(x)$ increases when $\bar{\varepsilon}$ decreases, so can define

$$\lim_{\bar{\varepsilon} \searrow 0} (u * \rho^+_{\bar{\varepsilon}})(x) = \sup_{\bar{\varepsilon} > 0} (u * \rho^+_{\bar{\varepsilon}})(x) =: u_0(x)$$

Hereby $u_0 : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

As a pointwise limit of increasing functions $u_0$ is increasing.

Thus if for some $x_0 \in \mathbb{R}$, $u_0(x_0) = \infty$, then $u_0(x) = \infty$ for all $x \geqq x_0$.

But this is impossible: if $\phi \in \mathcal{D}(\mathbb{R})$ and $\phi \geqq 0$, then

$$\langle u * \rho^+_\varepsilon, \phi \rangle = \langle u, \widetilde{\rho^+_\varepsilon} * \phi \rangle \underset{\varepsilon \searrow 0}{\longrightarrow} \langle u, \phi \rangle$$

and by Lebesgue's monotone convergence theorem also, as $\varepsilon \searrow 0$,

$$\langle u * \rho^+_\varepsilon, \phi \rangle = \int_{-\infty}^{\infty} u * \rho^+_\varepsilon \, \phi \, dx \nearrow \int_{-\infty}^{\infty} u_0 \phi \, dx.$$

Hence $\langle u, \phi \rangle = \int_{-\infty}^{\infty} u_0 \phi \, dx$

forcing $\int_{-\infty}^{\infty} u_0 \phi \, dx \in \mathbb{R}$ for all such $\phi$. Thus $u_0(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, and $u = u_0$, an increasing real-valued function. (For the final identification use also Problem 2 on Sheet 1.)

Conversely, assume $u \in \mathcal{D}'(\mathbb{R})$ can be represented by an increasing function: $u : \mathbb{R} \to \mathbb{R}$ increasing.

Then $\rho_\varepsilon * u \in C^\infty(\mathbb{R})$ and

$$(\rho_\varepsilon * u)(x) = \int_{-\infty}^{\infty} u(x-y)\rho_\varepsilon(y)\,dy$$

is increasing, so

$$0 \leqq (\rho_\varepsilon * u)'(x) = (\rho_\varepsilon * u')(x) \quad \forall x.$$

Take $\phi \in \mathcal{D}(\mathbb{R})$ with $\phi \geqq 0$. Then

$$0 \leqq \langle \rho_\varepsilon * u', \phi \rangle \xrightarrow[\varepsilon \searrow 0]{} \langle u', \phi \rangle. \quad \Box$$

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Using above result and Riesz-Markov: if $u : (a,b) \to \mathbb{R}$ is increasing, then

$$u' = \mu$$

a locally finite Borel measure on $(a,b)$.

$\boxed{\text{EX}}$ Let $\mu$ be a locally finite Borel measure on $(a,b)$. Put for $x_0 \in (a,b)$

$$u(x) = \begin{cases} \mu([x_0, x]) & \text{if } x \in [x_0, b) \\ -\mu((x, x_0)) & \text{if } x \in (a, x_0). \end{cases}$$

Then $u : (a,b) \to \mathbb{R}$ increasing (and right-continuous).

<u>Claim</u>
- $u$ continuous at $x \in (a,b)$ iff $\mu(\{x\}) = 0$,
- $u' = \mu$ in $\mathcal{D}'(a,b)$.

The first assertion is clear.

For second, let $\phi \in \mathcal{D}(a,b)$:

$$\langle u', \phi \rangle = -\langle u, \phi' \rangle =$$

$$+ \int_a^{x_0} \mu(x, x_0)\, \phi'(x)\, dx - \int_{x_0}^b \mu[x_0, x]\, \phi'(x)\, dx$$

$$= \int_a^{x_0} \int_a^b \mathbb{1}_{(x,x_0)}(t)\, \phi'(x)\, d\mu(t)\, \cancel{dx} - \int_{x_0}^b \int_a^b \mathbb{1}_{[x_0,x]}(t)\, \phi'(x)\, d\mu(t)\, dx$$

$$= \int_a^{x_0} \int_{(a,x_0)} \mathbb{1}_{(x,x_0)}(t)\, \phi'(x)\, d\mu(t)\, dx$$

$$- \int_{x_0}^{b} \int_{[x_0,b)} \mathbb{1}_{[x_0,x]}(t)\, \phi'(x)\, d\mu(t)\, dx$$

$$\overset{\text{Fubini}}{=} \int_{(a,x_0)} \int_a^{x_0} \mathbb{1}_{(x,x_0)}(t)\, \phi'(x)\, dx\, d\mu(t)$$

$$- \int_{[x_0,b)} \int_{x_0}^{b} \mathbb{1}_{[x_0,x]}(t)\, \phi'(x)\, dx\, d\mu(t)$$

$$= \int_{(a,x_0)} \int_a^{t} \phi'(x)\, dx\, d\mu(t) - \int_{[x_0,b)} \int_t^{b} \phi'(x)\, dx\, d\mu(t)$$

$$\overset{\text{FTC}}{=} \int_a^b \phi\, d\mu \,.$$

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By the constancy theorem all right-continuous functions $f : (a,b) \to \mathbb{C}$ so $f' = \mu$ in $\mathcal{D}'(a,b)$ are

$$f = u + c \,, \quad c \in \mathbb{C} .$$

$\boxed{\text{EX}}$ Let $f \in L^1_{loc}(a,b)$ and $x_0 \in (a,b)$.

Put $F(x) = \int_{x_0}^{x} f(t)\,dt$ , $x \in (a,b)$.

Then $F$ is continuous, but need not be (everywhere) differentiable.

Claim: $F' = f$ in $\mathcal{D}'(a,b)$

Calculate for $\phi \in \mathcal{D}(a,b)$:

$$\langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_a^b \int_{x_0}^{x} f(t)\,dt\, \phi'(x)\,dx$$

$$= +\int_a^{x_0} \int_x^{x_0} f(t)\,dt\, \phi'(x)\,dx - \int_{x_0}^{b} \int_{x_0}^{x} f(t)\,dt\, \phi'(x)\,dx$$

$$\overset{\text{Fubini}}{=} \int_a^{x_0} \int_a^{t} \phi'(x)\,dx\, f(t)\,dt - \int_{x_0}^{b} \int_t^{b} \phi'(x)\,dx\, f(t)\,dt$$

$$\overset{\text{FTC}}{=} \int_a^b \phi(t) f(t)\,dt \,.$$

This gives an easy alternative to $\text{FTC}^d$ for regular distributions.

$\boxed{\text{EX}}$ Find all $u \in \mathcal{D}'(\mathbb{R})$ satisfying

$$u' + au \geq 0 \text{ in } \mathcal{D}'(\mathbb{R}),$$

where $a \in C^\infty(\mathbb{R})$.

Assume $u$ satisfies the condition. By Riesz-Markov there exists a locally finite Borel measure $\mu$ on $\mathbb{R}$ so

$$u' + au = \mu$$

We put $A(x) = \int_0^x a(t)\,dt$, note $e^A$ is an integrating factor to get:

$$(e^A u)' = e^A \mu .$$

Then by a previous ex,

$$G(x) = \begin{cases} \int_{[0,x]} e^{A(t)}\,d\mu(t) & \text{if } x \geq 0 \\ -\int_{(x,0)} e^{A(t)}\,d\mu(t) & \text{if } x < 0 \end{cases}$$

is an increasing, right-continuous function so $G' = e^A \mu$

Thus $(e^{A}u - G)' = 0$, so by constancy theorem

$$u = e^{-A}G + ce^{-A}, \quad c \in \mathbb{C}.$$

Conversely any such function (note it is in particular right-continuous and locally bounded) satisfies the differential inequality:

$$u' + au = \mu \geqslant 0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Thus all such distributions on $\mathbb{R}$ are

$$e^{-A}(G + c)$$

where $c \in \mathbb{C}$ and $G$ is defined above corresponding to a measure $\mu$.
We say these $u$ are <u>sub-solutions</u> to the ODE $y' + ay = 0$ on $\mathbb{R}$.

The distributions $v \in \mathcal{D}'(\mathbb{R})$ satisfying
$v' + av \leqq 0$ in $\mathcal{D}'(\mathbb{R})$
are called <u>super-solutions</u> to the ODE $y' + ay = 0$ on $\mathbb{R}$.

Important classes of sub-solutions:

- Sub-solutions of $y'' = 0$:
  $u \in \mathcal{D}'(a,b)$ so $u'' \geqq 0$ in $\mathcal{D}'(a,b)$.
  These are called <u>convex</u>.
  (The corresponding super-solutions are called concave.)

•• Sub-solutions of $\Delta y = 0$:
  $u \in \mathcal{D}'(\Omega)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open $(n > 1)$
  so $\Delta u \geqq 0$ in $\mathcal{D}'(\Omega)$.
  These are called <u>subharmonic</u>
  (The corresponding super-solutions are called superharmonic.)