B4.1 Functional Analysis MT 2021: Problem Sheet 0 [not for handing in]

In the questions below the scalar field is assumed to be $\mathbb{R}$ for simplicity, but all results hold when the scalars are complex.

1. Let X be the vector space of real sequences (x_j) and define

$$\|(x_j)\| = \begin{cases} 0 & \text{if } x_j = 0 \text{ for all } j, \\ |x_{j_0}| & \text{if } j_0 = \min\{j \mid x_j \neq 0\}. \end{cases}$$

Show that the Triangle Inequality fails to hold, so that $\|\cdot\|$ is not a norm.

2. (i) Let X be a real inner product space and, for each $x \in X$, let $\|x\| = \langle x, x\rangle^{1/2}$. You may assume the fact that $\|\cdot\|$ does define a norm on X. Verify the Parallelogram Law: for all $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(ii) Consider the ∞ norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$ $(n \geqslant 2)$:

$$\|(x_1, \dots, x_n)\|_\infty = \sup_{1 \leqslant j \leqslant n} |x_j|.$$

By showing that the Parallelogram Law fails, prove that there is no inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that

$$\|x\|_\infty = \langle x, x\rangle^{1/2} \quad \text{for all } x \in \mathbb{R}^n.$$

3. Let X be a (real) vector space equipped with a norm $\|\cdot\|$. As usual we define a metric d on X by $d(x, y) = \|x - y\|$. For $x_0 \in X$ and $r > 0$, let

$$\mathrm{B}_r(x_0) = \{\, x \in X \mid \|x - x_0\| < r \,\} \qquad \text{(open ball)},$$
$$\overline{\mathrm{B}}_r(x_0) = \{\, x \in X \mid \|x - x_0\| \leqslant r \,\} \qquad \text{(closed ball)}.$$

[*The terminology was justified in the Metric Spaces course: it was shown that open balls are open sets and closed balls are closed sets.*]

(i) A subset C of X is **convex** if $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$ imply $\lambda x + (1-\lambda)y \in C$. Prove that $\mathrm{B}_r(x_0)$ and $\overline{\mathrm{B}}_r(x_0)$ are convex.

(ii) Prove that $\overline{\mathrm{B}}_r(x_0)$ is the closure of $\mathrm{B}_r(x_0)$.

(iii) Use (i) to show that $(x_1, x_2) \mapsto |x_1|^{1/2} + |x_2|^{1/2}$ does not define a norm on $\mathbb{R}^2$.

4. (i) Let X be a real normed space. Let $T\colon X \to \mathbb{R}$ be a linear map such that $|Tx| \leqslant \|x\|$ for all $x \in X$. Prove that T is continuous.

(ii) Let $X = \ell^p$, $1 \leqslant p \leqslant \infty$, equipped with the p-norm $\|x\|_p = \left(\sum_{j=1}^\infty |x_i|^p\right)^{1/p}$ respectively $\|x\|_\infty = \sup_j |x_j|$. Define $\pi_k\colon X \to \mathbb{R}$ by $\pi_k((x_j)) = x_k$ (for any $k \geqslant 1$). Check that each π_k is continuous.

(iii) Let $X = L^2([0,1])$ and define $T : X \to \mathbb{R}$ by $T(f) := \int_0^1 f dx$. Check that T is continuous. [*Hint: Use that by Hölder's inequality $\|fg\|_{L^1} \leqslant \|f\|_{L^2}\|g\|_{L^2}$ for every $f, g \in L^2([0,1])$.*]

(iv) Let X be as in (ii). Let (a_j) be a fixed sequence of real numbers and define

$$Y = \{\, (x_j) \in X \mid x_{2j} = a_j x_{2j-1} \text{ for all } j \geqslant 1 \,\}.$$

Check that Y is a subspace of X and, by writing Y as an intersection of closed sets involving maps π_k, or otherwise, show that Y is closed.

5. Let Y be a subspace of a normed space $(X, \|\cdot\|)$. Prove that Y is closed if and only if

$$\operatorname{dist}(x, Y) := \inf_{y \in Y} \|x - y\| > 0 \text{ for all } x \in X \setminus Y.$$