A model theoretic "Proof" for algebraic independence of e and π

Recall $x, y \in \mathbb{R}$ are called *algebraically independent* if $g(x,y) \neq 0$ for all non-zero polynomials $g(X,Y) \in \mathbb{Q}[X,Y]$

"**Claim**" e and π are algebraically independent

[This is, in fact, a big open problem in transcend. number theory]

"**Proof**": Let $\mathcal{L}_{ring} = \{+, \cdot ; 0, 1\}$ be the language of rings,

$RCF := Th(\langle \mathbb{R} ; +, \cdot ; 0, 1 \rangle)$ the $\mathcal{L}_{ring}$-theory of real closed fields

$p(x) := \{ n < x \mid n \in \mathbb{N} \}$ the 1-type for "infinitely large" elements

note that "$<$" is $\mathcal{L}_{ring}$ definable in models of RCF: $(x < y \leftrightarrow \exists z \neq 0 \; y = x + z^2)$.

Obs.: $p(x)$ is omitted in a model R of RCF iff R has an $\mathcal{L}_{ring}$-embedding into $\langle \mathbb{R} ; +, \cdot ; 0, 1 \rangle$

Now let c, d be new constants and let

$$\begin{aligned} T = RCF \; & \cup \; \{ r < c < r' \mid r, r' \in \mathbb{Q} \text{ with } r < e < r' \} \\ & \cup \; \{ s < d < s' \mid s, s' \in \mathbb{Q} \text{ with } s < \pi < s' \} \\ & \cup \; \{ g(c,d) \neq 0 \mid g \in \mathbb{Q}[X,Y] \setminus \{0\} \} \end{aligned}$$

Then T is finitely realizable (in $\mathbb{R}$), so, by compactness, T has a model.

Note that $p(x)$ is a non-principal type (for example, because it is omitted in $\mathbb{R}$)

Now let R be a model of T omitting p (by OTT).

Then, by Obs., R can be considered as subfield of $\mathbb{R}$ and c and d have to be interpreted as e and π resp." ∎

FLAW: If $g(e,\pi) = 0$ for some $g \in \mathbb{Q}[X,Y] \setminus \{0\}$ then $p(x)$ becomes a principal type of T, as c and d are infinitesimally close to e resp. π, but not equal; so $\frac{1}{|e-c|}$ or $\frac{1}{|\pi - d|} \gg 0$.