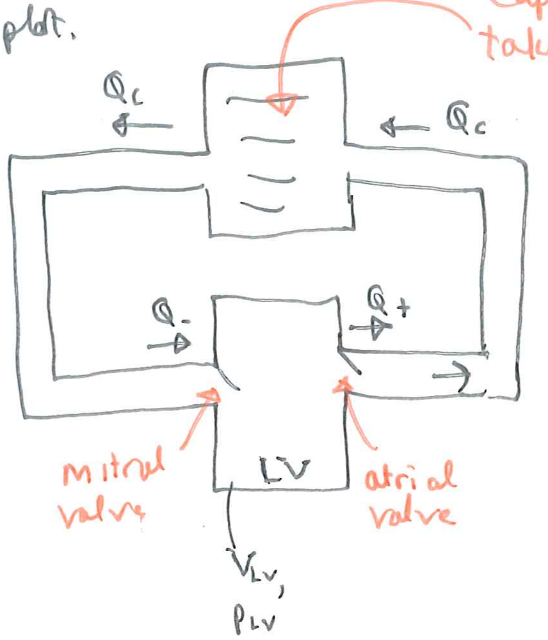
A simple mechanical model of the circulation

We would like to write down a mathematical model to explain the $p_{LV} - V_{LV}$ plot.

Capillaries – offer resistance but take up no volume.



veins
volume in veins : V_v
pressure : p_v
(assume no pressure loss from start to end)

arteries
volume in arteries : V_a
pressure : p_a

Q_+, Q_-, Q_c = blood flows
V_a, V_v, V_{LV} = compartment volumes

Blood is incompressible but blood vessels are compliant (so blood flows at different points along the network can differ from one another and change with time).

Conservation of blood :

$$\dot{V}_a = Q_+ - Q_-$$

$$\dot{V}_v = Q_c - Q_-$$

$$\dot{V}_{LV} = Q_- - Q_+$$

Resistances :

Capillaries occupy no volume but offer resistance :

$$Q_c = \frac{p_a - p_v}{R_c}$$

$$Q_+ = \frac{[p_{LV} - p_a]_+}{R_a}$$ the positive part

$$Q_- = \frac{[p_v - p_{LV}]_+}{R_v}$$

Compliance.

Increasing pressure distends blood vessels :

$$V_a = V_{a0} + C_a p_a$$

$$V_v = V_{v0} + C_v p_v$$

$$V_{LV} = V_{LV0} + C_{LV} p_{LV}$$

C = compliance $= \frac{1}{\text{elastance}}$

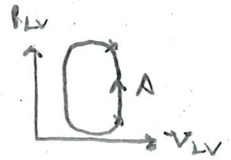
low C = tight, high C = loose

Writing the model in terms of just pressures :

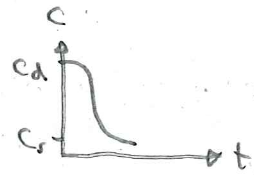
$$\frac{dp_a}{dt} = \frac{[p_{LV} - p_a]_+}{R_a C_a} - \frac{p_a - p_v}{R_c C_a} \quad (1)$$

$$\frac{dp_v}{dt} = \frac{p_a - p_v}{R_c C_v} - \frac{[p_v - p_{LV}]_+}{R_v C_v} \quad (2)$$

$$\frac{d}{dt}(C_{LV} p_{LV}) = \frac{[p_v - p_{LV}]_+}{R_v} - \frac{[p_{LV} - p_a]_+}{R_a} \quad (3)$$



(Both valves closed) (0.05 s)



A. Isovolumetric contraction

Initial conditions :

$p_{LV} < p_v \ll p_a$

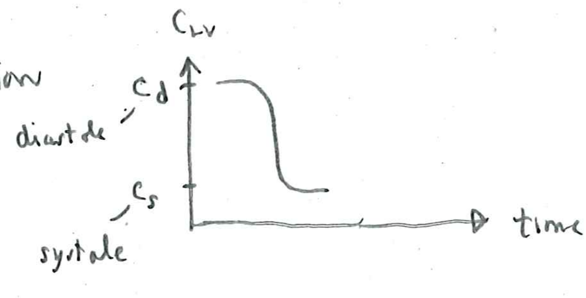
(1) $\Rightarrow$ $\frac{dp_a}{dt} = -\frac{p_a}{R_c C_a}$

But $R_c C_a \approx 1.8\,s$ and time of contraction $\approx 0.05\,s$ so $\boxed{p_a \approx \text{constant}}$ on this phase

Similarly, (2) $\Rightarrow$ $\boxed{p_v \approx \text{constant}}$ on this phase

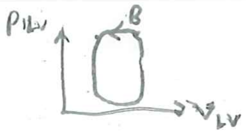
(3) $\Rightarrow$ $\frac{d}{dt}(C_{LV} p_{LV}) = 0$

Now



so C_{LV} falls $\therefore$ $\boxed{p_{LV} \text{ rises}}$

B. Ejection



(Atrial valve open) (0.3 s) (C_{LV} = constant = C_s) 77

$p_v \ll p_a < p_{LV}$ (because now p_{LV} is much higher)

① ⇒
$$\underbrace{\frac{dp_a}{dt}}_{0.3\,s} = \underbrace{\frac{p_{LV} - p_a}{R_a C_a}}_{0.09\,s} - \underbrace{\frac{p_a - p_v}{R_c C_a}}_{1.8\,s} \quad (*)$$

much smaller than the other two

⇒ $\boxed{p_{LV} \approx p_a}$

② ⇒
$$\underbrace{\frac{dp_v}{dt}}_{0.3\,s} = \underbrace{\frac{p_a - p_v}{R_c C_v}}_{60\,s} \Rightarrow \boxed{p_v \approx \text{constant}}$$

③ ⇒
$$\frac{d}{dt}(C_{LV} p_{LV}) = -\frac{p_{LV} - p_a}{R_a}$$

Now C_{LV} = constant so substituting this into (*) gives
= C_s

$$\frac{dp_a}{dt} = -\frac{C_s}{C_a}\frac{dp_a}{dt} - \frac{p_a - p_v}{R_c C_a}$$

$$(C_a + C_s)\frac{dp_a}{dt} = -\frac{p_a}{R_c} \quad \text{(since } p_a \gg p_v\text{)}$$

$$\boxed{p_a \propto \exp\left[\frac{-t}{R_c(C_a + C_s)}\right]}$$

0.3 s

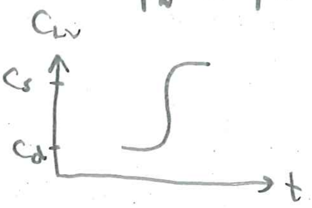
2.2 s

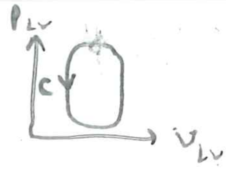
⇒ p_a falls by ~ 0.87 in this phase.

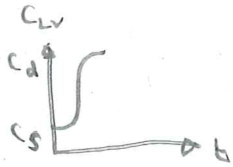
C. Isovolumetric relaxation

(Both valves closed) (0.08s)

$p_v < p_{LV} < p_a$







$\boxed{p_a = \text{constant}}$

$\boxed{p_v = \text{constant}}$

C_{LV} rises.

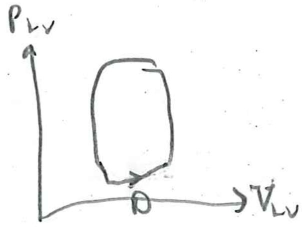
As before, $\frac{d}{dt}(C_{LV} p_{LV}) = 0$

So now $\boxed{p_{LV} \text{ falls}}$ until $p_{LV} = p_v$, and mitral valve opens to commence filling

D. Refilling

(Mitral valve open) (0.5s) | C_{LV} = constant $= C_d$)

$p_{LV} < p_v << p_a$



(1) $\Rightarrow$ $\frac{dp_a}{dt} = -\frac{p_a - p_v}{R_c C_a} \approx -\frac{p_a}{R_c C_a}$ since $p_a >> p_v$

(2) $\Rightarrow$ $\frac{dp_v}{dt} = \underbrace{\frac{p_a - p_v}{R_c C_v}}_{\text{neglect compared}} - \underbrace{\frac{p_v - p_{LV}}{R_v C_v}}_{\text{with 2nd term}}$ ($R_c C_v$: 60s; $R_v C_v$: 0.8s)

(3) $\Rightarrow$ $\frac{dp_{LV}}{dt} = \frac{p_v - p_{LV}}{C_d R_v}$ (since C_{LV} = constant $= C_d$)

(1) $\Rightarrow$ $\boxed{p_a \propto \exp\left[-\frac{t}{R_c C_a}\right]}$ $\Rightarrow$ p_a falls by ~ 0.76 in this phase

So total fall in p_a $\sim$ $\underset{\text{(ejection)}}{0.87} \times \underset{\text{(refilling)}}{0.76} = 0.66$ (120 mm Hg to 80 mm Hg)

① → ③ gives

$$\dot{p}_v - \dot{p}_{Lv} = -\left(\frac{1}{R_v C_v} + \frac{1}{R_v C_a}\right)(p_v - p_{Lv})$$

since (C_v = constant)

$$\boxed{p_v - p_{Lv} \propto \exp\left[-\frac{1}{R_v}\left(\frac{1}{C_v} + \frac{1}{C_a}\right)t\right]}$$