1(a) $W_{xxxx} = -\frac{\sigma}{B} W_{tt}$

Try $W(x,t) = f(x) e^{-i\omega t}$ (Re assumed)

$$f'''' = \frac{\sigma\omega^2}{B} f = \lambda^4 f$$

$f = f'' = 0$ at $x = 0, L$

$m^4 = \lambda^4$

$m = \pm\lambda, \pm i\lambda$

$$f(x) = A\cos(\lambda x) + B\sin(\lambda x) + \cancel{C\cosh}(\lambda x)$$
$$D\sinh(\lambda x)$$

1(b) $\boxed{\omega(k) = \sqrt{\frac{B}{\sigma}}\, k^2}$

1(c) $$W(x,t) = \frac{a}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|k|}\cos(\omega(k)t)\, e^{ikx}\, dk \qquad (x = Vt)$$

$$= \frac{a}{4\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|k|}\left(e^{i(kV-\omega(k))t} + e^{i(-kV+\omega(k))t}\right) dk$$

[$k \mapsto -k$ in 2nd term]

$$\boxed{W(x,t) = \frac{a}{2\pi}\mathrm{Re}\int_{-\infty}^{\infty} \underbrace{e^{-\varepsilon|k|}}_{f}\, e^{i\underbrace{(kV-\omega(k))}_{\psi}t}\, dk}$$

plug into formula...

2(a) $\nabla\cdot(\rho\underline{u}) = 0$, $\rho(\underline{u}\cdot\nabla)\underline{u} = -\nabla p$, $p = P(\rho)$

$\underline{u} = U\underline{e}_x + \nabla\phi$, $p = p_0 + p'$, $\rho = \rho_0 + \rho'$

$p_0 + p' = P(\rho_0 + \rho')$

$\Rightarrow \boxed{p' = c_0^2\rho'}$ where $\boxed{c_0^2 = P'(\rho_0)}$

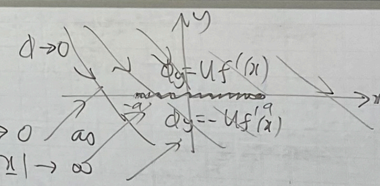
Linearised momentum eq^n: $\rho_0 U \frac{\partial}{\partial x}\nabla\phi = -\nabla p'$

$\Leftrightarrow \nabla\left(p' + \rho_0 U \frac{\partial\phi}{\partial x}\right) = 0$

$\therefore \boxed{p' = -\rho_0 U \frac{\partial\phi}{\partial x}}$ (everything $\to 0$ at ∞)

plug into cons. of mass... $\boxed{M = U/c_0}$

$$(M^2-1)\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial y^2}$$

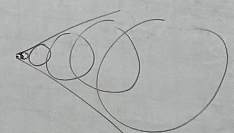


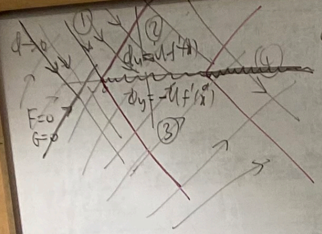
① $M<1$ elliptic $\nabla\phi \to 0$ as $|\underline{x}| \to \infty$

② $M>1$ hyperbolic $\phi \to 0$ as $x \to -\infty$

We can only specify B/Cs once on each characteristic

By causality, impose B/Cs at upstream end of domain





$$\phi(x,y) = F(x-By) + G(x+By)$$

$$B = \sqrt{M^2-1} > 0$$

① $\phi = 0$

② Still $G=0$

$$\phi(x,y) = F(x-By)$$

$$-BF'(x) = Uf'(x)$$

$$F(x) = -\frac{U f(x)}{B} + \text{const.}$$

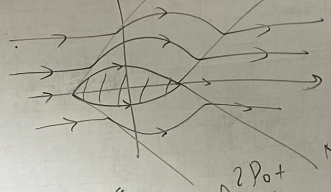
$$\boxed{② \ \phi(x,y) = -\frac{U}{B} f(x-By)} + \cancel{\frac{U}{B} f(-a)}$$

given $f(-a) = 0$

$$\boxed{③ \ \phi(x,y) = -\frac{U}{B} f(x+By)}$$

NB in this case $f(\pm a) = 0 \rightarrow$ no circulation: ϕ is continuous everywhere. In general, ϕ could be discontinuous across $x > a$.

$\eta(1)$



NB $P = P_0 + P'$

P_0 term integrates to zero

$$D = \int_{-a}^{a} f'(x) \left[\overset{2P_0 +}{\wedge} P'(x,0-) + P'(x,0+) \right] dx$$

$$= -\rho_0 U \int_{-a}^{a} f'(x) \left[\phi_x(x,0-) + \phi_x(x,0+) \right] dx$$

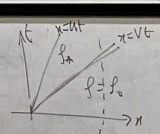
$$= \frac{2\rho_0 U^2}{B} \int_{-a}^{a} \underbrace{f'(x)^2}_{= \frac{4b^2x^2}{a^4}} dx = \frac{16\rho_0 b^2}{3a} \frac{U^2}{B}$$

$$B = \sqrt{\frac{U^2}{c_0^2} - 1}$$

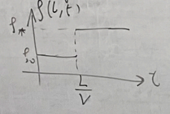
$$= \frac{16\rho_0 b^2}{3a} c_0^2 \left(B + \frac{1}{B} \right)$$

minimised when $B = 1$

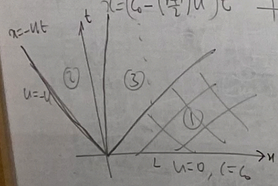
$\Leftrightarrow U = \sqrt{2}\, c_0$



$$\rho_* = \frac{\rho_0 V}{V-U} > \rho_0$$



$x = \left(c_0 - \left(\frac{\gamma+1}{2}\right)U\right)t$



① $u = 0, \quad c = c_0$

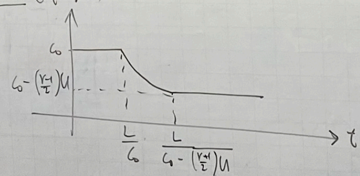
② $u = -U, \quad c = c_0 - \left(\frac{\gamma-1}{2}\right)U > 0$

given $U < \frac{2c_0}{\gamma+1}$

③ $u = -\frac{2}{\gamma-1}\left(c_0 - \frac{x}{t}\right)$

$$c = c_0 - \left(\frac{\gamma-1}{\gamma+1}\right)\left(c_0 - \frac{x}{t}\right)$$

$U < \frac{2c_0}{\gamma+1}$ $c(L,t)$



$U > \frac{2c_0}{\gamma+1}$ $c(L,t)$