Model solutions and marking scheme for B4.3. December 2019

Question 1: (a)

- φ is a test function on $\mathbb{R}$, $\varphi \in \mathcal{D}(\mathbb{R})$, if $\varphi\colon \mathbb{R} \to \mathbb{C}$ is a C^∞ function with compact support, $\mathrm{supp}(\varphi) := \overline{\{x \in \mathbb{R} : \varphi(x) \neq 0\}}$.
- $\varphi_j \to \varphi$ in $\mathcal{D}(\mathbb{R})$ if for some compact $K \subset \mathbb{R}$ we have $\mathrm{supp}(\varphi_j)$, $\mathrm{supp}(\varphi) \subseteq K$ for all j and $\varphi_j^{(s)} \to \varphi^{(s)}$ uniformly on $\mathbb{R}$ for each $s \in \mathbb{N}_0$.
- u is a distribution on $\mathbb{R}$, $u \in \mathcal{D}'(\mathbb{R})$, if $u\colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ is linear and $\mathcal{D}$-continuous: if $\varphi_j \to \varphi$ in $\mathcal{D}(\mathbb{R})$, then $\langle u, \varphi_j\rangle \to \langle u, \varphi\rangle$ as $j \to \infty$.

[**Marks 1+1+1. Bookwork**]

A linear functional $u\colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ is a distribution on $\mathbb{R}$ iff it has the boundedness property: if $K \subset \mathbb{R}$ is compact, then there exist constants $c = c_K \geqslant 0$, $m = m_K \in \mathbb{N}_0$ so

$$\bigl|\langle u, \phi\rangle\bigr| \leqslant c \sum_{s=0}^{m} \sup_K \bigl|\phi^{(s)}\bigr| \tag{1}$$

holds for all $\phi \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(\phi) \subseteq K$.

Proof. Assume first, $u \in \mathcal{D}'(\mathbb{R})$ but that u hasn't got the boundedness property (1): there exists a compact set $K \subset \mathbb{R}$ so that for each $c = m = j \in \mathbb{N}$ we can find $\varphi_j \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(\varphi_j) \subseteq K$ and $|\langle u, \varphi_j\rangle| > j \sum_{s=0}^{j} \sup |\varphi_j^{(s)}|$. Put $\lambda_j := |\langle u, \varphi_j\rangle| > 0$. If $\psi_j := \varphi_j/\lambda_j$, then $\psi_j \in \mathcal{D}(\mathbb{R})$ have supports in K and $1/j > \sum_{s=0}^{j} \sup |\psi_j^{(s)}|$ for all j. It follows that $\psi_j \to 0$ in $\mathcal{D}(\mathbb{R})$ and hence, by $\mathcal{D}$-continuity of u, that $1 = |\langle u, \psi_j\rangle| \to 0$, a contradiction proving that u must have the boundedness property.

Next, if u is linear and has the boundedness property, then for $\varphi_j \to \varphi$ in $\mathcal{D}(\mathbb{R})$ as above, we find constants $c = c_K$, $m = m_K$ so (1) holds for $\phi = \varphi_j - \varphi$. The $\mathcal{D}$-continuity is then a consequence of linearity. □

[**Marks 1+2. Bookwork**]

$u \in \mathcal{D}'(\mathbb{R})$ has order at most m if we can take $m_K = m$ for all compact $K \subset \mathbb{R}$ in (1). [**1 mark. Bookwork**]

Assume u has order at most m. Fix a compact $K \subset \mathbb{R}$ and let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subseteq K$. Then from $\mathrm{supp}(\varphi') \subseteq \mathrm{supp}(\varphi)$ we get by (1):

$$\begin{aligned}|\langle u', \varphi\rangle| = |\langle u, -\varphi'\rangle| &\leqslant c \sum_{s=0}^{m} \sup |\varphi^{(s+1)}| \\ &\leqslant c \sum_{s=0}^{m+1} \sup |\varphi^{(s)}|\end{aligned}$$

and u' has order at most $m+1$ as required. **[1 mark. Seen before]**
(b) Clearly $E(\varphi)$ is well-defined and by FTC is C^1 with $E(\varphi)' = \varphi - \rho\int_{\mathbb{R}}\varphi\,\mathrm{d}t \in \mathrm{C}^\infty$ so $E(\varphi) \in \mathrm{C}^\infty(\mathbb{R})$. Take $a > 1$ so large that $\mathrm{supp}(\varphi) \subseteq [-a,a]$. Then $\mathrm{supp}(E(\varphi)) \subseteq [-a,a]$ too: clearly $E(\varphi) = 0$ on $(-\infty,a]$ and if $x \geqslant a$, then

$$E(\varphi)(x) = \int_{-\infty}^{x}\big(\varphi(s) - \rho(s)\int_{\mathbb{R}}\varphi\big)\,\mathrm{d}s$$
$$= \int_{\mathbb{R}}\big(\varphi(s) - \rho(s)\int_{\mathbb{R}}\varphi\big)\,\mathrm{d}s = 0.$$

Therefore $\mathrm{supp}(E(\varphi)) \subseteq [-a,a]$, so $E(\varphi) \in \mathcal{D}(\mathbb{R})$ and $E\colon \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$. It is clear that E is linear. We check $\mathcal{D}$-continuity: Assume $\phi_j \to \phi$ in $\mathcal{D}(\mathbb{R})$. Take $a > 1$ containing all supports so that, by above argument, the supports of $E(\phi_j)$, $E(\phi)$ are also contained in $[-a,a]$. Next, we estimate

$$\sup_{\mathbb{R}}\big|E(\phi_j) - E(\phi)\big| \leqslant 2\int_{-a}^{a}|\phi_j - \phi|\,\mathrm{d}x \leqslant 4s\sup_{\mathbb{R}}|\phi_j - \phi| \to 0,$$

and for $k \in \mathbb{N}$,

$$\sup_{\mathbb{R}}\Big|\big(E(\phi_j) - E(\phi)\big)^{(k)}\Big| = \sup_{\mathbb{R}}\Big|\phi_j^{(k-1)} - \phi^{(k-1)} + \int_{\mathbb{R}}(\phi_j - \phi)\,\mathrm{d}s\rho^{(k-1)}\Big|$$
$$\leqslant \sup_{\mathbb{R}}\big|\phi_j^{(k-1)} - \phi^{(k-1)}\big| + \int_{-a}^{a}|\phi_j - \phi|\,\mathrm{d}s\sup_{\mathbb{R}}|\rho^{(k-1)}| \to 0.$$

Thus E is linear and $\mathcal{D}$-continuous, and it is then clear that $U = P(u) \in \mathcal{D}'(\mathbb{R})$. By FTC we see that $E(\phi') = \phi$ holds for all $\phi \in \mathcal{D}(\mathbb{R})$, so $\langle U', \phi\rangle = -\langle u, \phi'\rangle = \langle u', \phi\rangle$, that is, $U' = u$. **[5 marks. Variant of bookwork.]**
(c) Assume $u \in \mathcal{D}'_m(\mathbb{R})$ for an $m \geqslant 1$. Fix a compact set $K \subset \mathbb{R}$. Take $a > 1$ so $K \subset [-a,a]$ and find $c = c_a \geqslant 0$ so

$$\big|\langle u, \phi\rangle\big| \leqslant c\sum_{k=0}^{m}\sup_{\mathbb{R}}|\phi^{(k)}|, \quad \forall\,\phi \in \mathcal{D}([-a,a]).$$

If $\phi \in \mathcal{D}(K)$, then by (b) $E(\phi) \in \mathcal{D}([-a,a])$ and using also the bounds from (b), $\sup|E(\phi)| \leqslant 4a\sup|\phi|$ and for $k \geqslant 1$,

$$\sup|E(\phi)^{(k)}| \leqslant \sup|\phi^{(k-1)}| + 2a\sup|\rho^{(k-1)}|\sup|\phi|.$$

If $C = 4a + 2a(m+1)\sup\{|\rho^{(k-1)}(x)| : k \leqslant m, x \in \mathbb{R}\}$, then

$$\big|\langle U, \phi\rangle\big| \leqslant C\sum_{k=0}^{m-1}\sup_{\mathbb{R}}|\phi^{(k)}|, \quad \forall\,\phi \in \mathcal{D}(K).$$

Thus U has order at most $m-1$.

If u has order 0, then $P(u)$ need not be continuous: $u = \delta_0$ clearly has order 0 and results in $P(u) = \tilde{H} - \int_{-\infty}^{0} \rho \, \mathrm{d}s$ that is discontinuous at 0. Put $f = P(u)$ so that from HINT $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ and if $v = P^2(u)$, then $v' = f$ in $\mathcal{D}'(\mathbb{R})$. But

$$F(x) = \int_0^x f(t) \, \mathrm{d}t, \; x \in \mathbb{R},$$

is a $\mathrm{W}^{1,1}_{\mathrm{loc}}$ function, that in particular is continuous, and $F' = f$ in $\mathcal{D}'(\mathbb{R})$ too. By the constancy theorem, $v = f + c$ for some constant c, so v is continuous too. **[3+1+4 marks. New example**

(d) A compactly supported distribution has finite order, say $v \in \mathcal{D}'_m(\mathbb{R})$. By (c), $P^m(v)$ has order 0 and so $f := P^{m+2}(v) \in \mathrm{C}(\mathbb{R})$. Also we have $f^{(m+2)} = v$ as required.

Next, suppose $f \in \mathrm{C}_c(\mathbb{R})$ and $f^{(n)} = v$ for some $n \geqslant 1$. Take $\chi \in \mathcal{D}(\mathbb{R})$ so $\chi = 1$ near the supports of v, f. Then $\langle v, x^{n-1} \rangle = \langle v, x^{n-1}\chi \rangle$ where we have $x^{n-1}\chi \in \mathcal{D}(\mathbb{R})$. Now

$$\langle f^{(n)}, x^{n-1}\chi \rangle = (-1)^n \langle f, \left(x^{n-1}\chi\right)^{(n)} \rangle = 0$$

since $(x^{n-1}\chi)^{(n)} = 0$ near the support of f, as required.
[**2+2 marks. First result treated on Problem sheet by different method, the second is a new example.**]

Question 2: (a) The constancy theorem: Let $u \in \mathcal{D}'(\mathbb{R})$ and assume that $u' = 0$ in $\mathcal{D}'(\mathbb{R})$. Then u is constant.

Proof. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$ and put $u_\varepsilon = \rho_\varepsilon * u$. Then $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$, $u_\varepsilon \to u$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \searrow 0$ and $u_\varepsilon' = \rho_\varepsilon * u' = 0$ in the usual sense on $\mathbb{R}$. Then by the usual constancy theorem, $u_\varepsilon = c_\varepsilon$ on $\mathbb{R}$ for some constant $c_\varepsilon \in \mathbb{C}$. Now

$$c_\varepsilon = c_\varepsilon \int_{\mathbb{R}} \rho \,\mathrm{d}x = \langle u_\varepsilon, \rho \rangle \to \langle u, \rho \rangle$$

as $\varepsilon \searrow 0$. Consequently, if $c = \langle u, \rho \rangle$, then we have for $\varphi \in \mathcal{D}(\mathbb{R})$ as $\varepsilon \searrow 0$: $\langle u_\varepsilon, \varphi \rangle \to \langle u, \varphi \rangle$ and also $\langle u_\varepsilon, \varphi \rangle = c_\varepsilon \int_{\mathbb{R}} \varphi \,\mathrm{d}x \to c \int_{\mathbb{R}} \varphi \,\mathrm{d}x$ and therefore $u = c$. □ **[1+2 marks. Bookwork]**

(i) u is in $\mathrm{W}^{k,1}(\mathbb{R}^n)$ if the distributional derivatives $\partial^\alpha u \in \mathrm{L}^1(\mathbb{R}^n)$ for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant k$. **[1 mark. Bookwork]**

(ii) Assume $x_j \to x$ in $\mathbb{R}$. Estimating

$$|F(x_j) - F(x)| \leqslant \begin{cases} \int_x^{x_j} |f| & \text{if } x < x_j \\ \int_{x_j}^{x} |f| & \text{if } x \geqslant x_j. \end{cases}$$

it follows from DCT that $F(x_j) \to F(x)$ and thus that F is continuous. In particular F is a regular distribution and for $\varphi \in \mathcal{D}(\mathbb{R})$ we get from Fubini and FTC:

$$\begin{aligned} \langle F', \varphi \rangle = -\langle F, \varphi' \rangle &= -\int_{-\infty}^{\infty} F(x)\varphi'(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{0}\int_{x}^{0} f(t)\varphi'(x)\,\mathrm{d}t\,\mathrm{d}x - \int_{0}^{\infty}\int_{0}^{x} f(t)\varphi'(x)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{-\infty}^{0}\int_{-\infty}^{t} \varphi'(x)\,\mathrm{d}x f(t)\,\mathrm{d}t - \int_{0}^{\infty}\int_{t}^{\infty} \varphi'(x)\,\mathrm{d}x f(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{0} \varphi(t)f(t)\,\mathrm{d}t - \int_{0}^{\infty} -\varphi(t)f(t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} f(t)\varphi(t)\,\mathrm{d}t \end{aligned}$$

as required. Hence if $u \in \mathrm{W}^{1,1}(\mathbb{R})$, so u, $u' \in \mathrm{L}^1(\mathbb{R})$, we have that $U(x) = \int_0^x u'(t)\,\mathrm{d}t$ is a continuous function with $U' = u'$ in $\mathcal{D}'(\mathbb{R})$. Consequently, $(U-u)' = 0$ in $\mathcal{D}'(\mathbb{R})$ so that $u = U + c$ for some constant c by the constancy

theorem and $U+c$ is a continuous representative for u. [**1+3+1 marks. Seen before**]
(b) From two applications of FTC we have for all $(x,y)\in\mathbb{R}^2$:

$$\begin{aligned}|\varphi(x,y)| &= \left|\int_{-\infty}^{x}\int_{-\infty}^{y}\partial_1\partial_2\varpi(s,t)\,\mathrm{d}t\,\mathrm{d}s\right|\\ &\leqslant \int_{-\infty}^{x}\int_{-\infty}^{y}\left|\partial_1\partial_2\varphi(s,t)\right|\mathrm{d}t\,\mathrm{d}t\\ &\leqslant \|\partial_1\partial_2\varphi\|_1\end{aligned}$$

whereby the desired bound follows. [**2 marks. New ex**] Next, put $\chi_n := \rho * \mathbf{1}_{[-n-1,n+1]}$. Then clearly $\chi_n\in\mathcal{D}(\mathbb{R})$, $\chi_n=1$ near $[-n,n]$, $\operatorname{supp}(\chi_n)\subseteq[-n-2,n+2]$ and $|\chi_n^{(s)}| = |\rho^{(s)} * \mathbf{1}_{[-n-1,n+1]}| \leqslant \|\rho^{(s)}\|_1 =: c_s$. If $\varphi=\psi\chi_n\otimes\chi_n$, then $\varphi\in\mathcal{D}(\mathbb{R}^2)$ and from the above we get for $(x,y)\in[-n,n]^2$ by use of the Leibniz rule and triangle inequality:

$$\begin{aligned}|\psi(x,y)| &\leqslant \|\partial_1\partial_2\varphi\|_1\\ &= \|\partial_1\partial_2\psi\chi_n\otimes\chi_n + \partial_2\psi\chi_n'\otimes\chi_n + \partial_1\psi\chi_n\otimes\chi_n' + \psi\chi_n'\otimes\chi_n'\|_1\\ &\leqslant \|\partial_1\partial_2\psi\|_1 + c_1\|\partial_1\psi\|_1 + c_1\|\partial_2\psi\|_1 + c_1^2\|\psi\|_1\\ &\leqslant c\big(\|\psi\|_1 + \|\partial_1\psi\|_1 + \|\partial_2\psi\|_1 + \|\partial_1\partial_2\psi\|_1\big)\end{aligned}$$

for $c=1+c_1^2$. The conclusion follows because the upper bound is independent of n. [**4 marks. New ex**]

Next, if $u\in\mathrm{W}^{2,1}(\mathbb{R}^2)$, then we let $u_\varepsilon=\rho_\varepsilon * u$, note that $u_\varepsilon\in\mathrm{C}^\infty(\mathbb{R}^2)$ and that for a multi-index $\alpha\in\mathbb{N}_0^2$ of length at most 2, $\partial^\alpha u_\varepsilon = \rho_\varepsilon * \partial^\alpha u$ and $\|\partial^\alpha u_\varepsilon\|_1\leqslant\|\partial^\alpha u\|_1$ so that $u_\varepsilon\in\mathrm{W}^{2,1}(\mathbb{R}^2)$ too. Also, $\|\partial^\alpha u_\varepsilon - \partial^\alpha u\|_1\to 0$ as $\varepsilon\searrow 0$. For $\varepsilon_1>0$, $\varepsilon_2>0$ we take $\psi = u_{\varepsilon_1}-u_{\varepsilon_2}\in\mathrm{W}^{2,1}(\mathbb{R}^2)\cap\mathrm{C}^\infty(\mathbb{R}^2)$ in (1) whereby we see that u_ε is a uniform Cauchy family for $\varepsilon\searrow 0$ and so is uniformly convergent. It follows that u has a continuous representative. [**5 marks. New ex**]
(c) The answer to question (i) is no: Let $u=\partial_1\partial_2 f$, where f is Ornstein's function. Then u has order 1 and since $\partial_1 u = \partial_2\partial_1^2 f$, $\partial_2 u = \partial_1\partial_2^2 f$, where $\partial_1^2 f$ and $\partial_2^2 f$ are regular distribtions, so in particular of order 0, it follows that both $\partial_1 u$ and $\partial_2 u$ have orders at most 1. [**2 marks. New ex**] The answer to question (ii) is no: Take $\varphi_\varepsilon=\rho_\varepsilon * f$, where $\big(\rho_\varepsilon\big)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}^2$. Then $\operatorname{supp}(\varphi_\varepsilon)\subseteq\operatorname{supp}(f)+\overline{B_\varepsilon(0)}$ so $\varphi_\varepsilon\in\mathcal{D}(\mathbb{R}^2)$ and if we assume that we have a constant c so $\|\partial_1\partial_2\varphi\|_1\leqslant c\big(\|\partial_1^2\varphi\|_1+\|\partial_2^2\varphi\|_1\big)$ for all $\varphi\in\mathcal{D}(\mathbb{R}^2)$, then a contradiction is obtained. Indeed, note $\partial_1\partial_2\varphi_\varepsilon = \rho_\varepsilon * \partial_1\partial_2 f$ and $\partial_j^2\varphi_\varepsilon = \rho_\varepsilon * \partial_j^2 f\to\partial_j^2 f$ in $\mathrm{L}^1(\mathbb{R}^2)$ as $\varepsilon\searrow 0$, hence taking

$\varphi = \varphi_{\varepsilon_1} - \varphi_{\varepsilon_2}$ for ε_1, $\varepsilon_2 > 0$ above we deduce that the family $\partial_1\partial_2\varphi_\varepsilon$ is Cauchy in $\mathrm{L}^1(\mathbb{R}^2)$, hence convergent in $\mathrm{L}^1(\mathbb{R}^2)$, as $\varepsilon \searrow 0$. But since also $\partial_1\partial_2\varphi_\varepsilon \to \partial_1\partial_2 f$ in $\mathcal{D}'(\mathbb{R}^2)$ as $\varepsilon \searrow 0$, and L^1 convergence implies $\mathcal{D}'$ convergence a contradiction is reached. [**3 marks. New ex**]

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Question 3: (a) The distributional derivative u' of $u \in \mathcal{D}'(\mathbb{R})$ is defined as $\langle u', \varphi\rangle := -\langle u, \varphi'\rangle$ for $\varphi \in \mathcal{D}(\mathbb{R})$. The distributional product au of $u \in \mathcal{D}'(\mathbb{R})$ and $a \in \mathrm{C}^\infty(\mathbb{R})$ is defined as $\langle au, \varphi\rangle := \langle u, a\varphi\rangle$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Since φ', $a\varphi \in \mathcal{D}(\mathbb{R})$ it is clear that u', $au \colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ are well-defined and linear. We check that they are also $\mathcal{D}$-continuous: Let $\varphi_j \to \varphi$ in $\mathcal{D}(\mathbb{R})$, that is $\mathrm{supp}(\varphi_j)$, $\mathrm{supp}(\varphi) \subseteq K$ for some fixed compact $K \subset \mathbb{R}$ and $(\varphi_j - \varphi)^{(s)} \to 0$ uniformly on $\mathbb{R}$ for each $s \in \mathbb{N}_0$. Clearly $\mathrm{supp}(\varphi_j') \subseteq \mathrm{supp}(\varphi_j)$ and $(\varphi_j' - \varphi')^{(s)} \to 0$ uniformly on $\mathbb{R}$, so $\langle u', \varphi_j\rangle \to \langle u', \varphi\rangle$. Clearly also $\mathrm{supp}(a\varphi_j) \subseteq \mathrm{supp}(\varphi_j)$ and by the Leibniz rule $(a\varphi_j - a\varphi)^{(s)} \to 0$ uniformly on $\mathbb{R}$, so $\langle au, \varphi_j\rangle \to \langle au, \varphi\rangle$.

Leibniz rule on $\mathcal{D}'$: $(au)' = a'u + au'$ when $a \in \mathrm{C}^\infty(\mathbb{R})$, $u \in \mathcal{D}'(\mathbb{R})$.

Proof. For $\varphi \in \mathcal{D}(\mathbb{R})$ we check the actions of the distributions using the previous definitions and linearity of u:

$$\begin{aligned}\langle (au)', \varphi\rangle - \langle a'u + au', \varphi\rangle &= -\langle au, \varphi'\rangle - \langle u, a'\varphi\rangle - \langle u', a\varphi\rangle \\ &= -\langle u, a\varphi' + a'\varphi\rangle + \langle u, (a\varphi)'\rangle \\ &= 0\end{aligned}$$

where the last equality follows from the usual Leibniz' rule. □

We have $\log|x| \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ so it is a regular distribution and for $\varphi \in \mathcal{D}(\mathbb{R})$ we get by integration by parts:

$$\begin{aligned}\langle \tfrac{\mathrm{d}}{\mathrm{d}x}\log|x|, \varphi\rangle &= -\int_{-\infty}^{\infty} \log|x|\varphi'(x)\,\mathrm{d}x \\ &= \lim_{\varepsilon \searrow 0}\left(-\int_{-\infty}^{-\varepsilon} - \int_{\varepsilon}^{\infty}\right)\log|x|\varphi'(x)\,\mathrm{d}x \\ &= \lim_{\varepsilon \searrow 0}\left(-\big[\log|x|\varphi(x)\big]_{x\to-\infty}^{x=-\varepsilon} + \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x - \big[\log|x|\varphi(x)\big]_{x=\varepsilon}^{x\to\infty} + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x}\,\mathrm{d}x\right) \\ &= \lim_{\varepsilon \searrow 0}\left(\log\varepsilon\big(\varphi(\varepsilon) - \varphi(-\varepsilon)\big) + \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x}\,\mathrm{d}x\right) \\ &= \langle \mathrm{pv}(\frac{1}{x}), \varphi\rangle\end{aligned}$$

where we used that $\log\varepsilon\big(\varphi(\varepsilon) - \varphi(-\varepsilon)\big) = \varepsilon\log\varepsilon\frac{\varphi(\varepsilon)-\varphi(-\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \searrow 0$. It follows in particular that $\mathrm{pv}\big(\frac{1}{x}\big)$ is a distribution on $\mathbb{R}$.

[**3+3 marks. Bookwork and Example known from Problem Sheets**]

(b) Fix $\varphi \in \mathcal{D}(\mathbb{R})$. We start by checking that $\langle \mathrm{fp}\big(\frac{1}{x^2}\big), \varphi \rangle$ is well-defined, and have by Taylor's formula

$$\frac{\varphi(x) - \varphi(0) - \varphi'(0)x}{x^2} \to \frac{\varphi''(0)}{2} \quad \text{as } x \to 0$$

so the integral over any interval $(-r, r)$ with $r > 0$ exists. If we take $r > 1$ so large that φ is supported in $(-r, r)$ the remaining part of the integrand is $\varphi(0)x^{-2}$ which is clearly integrable over $\mathbb{R} \setminus (-r, r)$. Thus the expression is well-defined. Next, we calculate using integration by parts:

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{pv}\big(\frac{1}{x}\big), \varphi \right\rangle &= -\left\langle \mathrm{pv}\big(\frac{1}{x}\big), \varphi' \right\rangle \\
&= \lim_{\varepsilon \searrow 0} -\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi'(x)}{x}\,\mathrm{d}x \\
&= \lim_{\varepsilon \searrow 0} \left(\frac{\varphi(-\varepsilon) + \varphi(\varepsilon)}{\varepsilon} - \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi(x)}{x^2}\,\mathrm{d}x \right) \\
&= \lim_{\varepsilon \searrow 0} \left(\frac{\varphi(-\varepsilon) + \varphi(\varepsilon) - 2\varphi(0)}{\varepsilon} - \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi(x) - \varphi(0)}{x^2}\,\mathrm{d}x \right)
\end{aligned}$$

Since $x \mapsto \frac{\varphi'(0)}{x}\mathbf{1}_{(-1,1)}(x)$ is an odd function we have for each $\varepsilon \in (0, 1)$ that

$$\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi(x) - \varphi(0)}{x^2}\,\mathrm{d}x = \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi(x) - \varphi(0) - \varphi'(0)x\mathbf{1}_{(-1,1)}(x)}{x^2}\,\mathrm{d}x$$

and since we in particular know that the finite part integral is well-defined we have shown that

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{pv}\big(\frac{1}{x}\big), \varphi \right\rangle &= \lim_{\varepsilon \searrow 0} \frac{\varphi(-\varepsilon) + \varphi(\varepsilon) - 2\varphi(0)}{\varepsilon} - \left\langle \mathrm{fp}\big(\frac{1}{x^2}\big), \varphi \right\rangle \\
&= -\left\langle \mathrm{fp}\big(\frac{1}{x^2}\big), \varphi \right\rangle.
\end{aligned}$$

Hence it follows in particular that $\mathrm{fp}\big(\frac{1}{x^2}\big)$ is a distribution. Next, we calculate for $\varphi \in \mathcal{D}(\mathbb{R})$, since $x^2\varphi(x) = \big(x^2\varphi(x)\big)' = 0$ at $x = 0$,

$$\langle x^2\mathrm{fp}\big(\tfrac{1}{x^2}\big), \varphi \rangle = \langle \mathrm{fp}\big(\tfrac{1}{x^2}, x^2\varphi \rangle = \int_{\mathbb{R}} \varphi\,\mathrm{d}x$$

as required. [**1+4+2 marks. Seen before on Problem Sheet**]

(c) We have for $\varepsilon > 0$,

$$\big(x + \mathrm{i}\varepsilon\big)^{-2} = \tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{Log}\,(x + \mathrm{i}\varepsilon) \quad \text{in } \mathcal{D}'(\mathbb{R})$$

and classically on $\mathbb{R}$. Now by inspection, as $\varepsilon \searrow 0$,

$$\mathrm{Log}\,(x+\mathrm{i}\varepsilon) \to \log|x| + \mathrm{i}\pi H(-x) \;\text{ in }\; \mathcal{D}'(\mathbb{R})$$

and so by $\mathcal{D}'$ continuity of differentiation,

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\big(x+\mathrm{i}\varepsilon\big)^{-2} =& \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(\log|x| + \mathrm{i}\pi H(-x)\Big)\\ =& \mathrm{fp}\Big(\tfrac{1}{x^2}\Big) - \mathrm{i}\pi\delta_0'.\end{aligned}$$

[**5 marks. Seen related ex before on Problem Sheet**]

(d) The equation is an inhomogeneous linear equation, and so GS can be found as the sum of a PS and GS to the corresponding homogeneous equation. In order to find a PS we observe that $\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$ for $x \neq 0$, 1. It follows from the result in (c) and its version shifted to 1, that $-\mathrm{pv}\big(\frac{1}{x}\big) - \mathrm{fp}\big(\frac{1}{x^2}\big) + \mathrm{pv}\big(\frac{1}{x-1}\big)$ is a PS to the equation in $\mathcal{D}'(\mathbb{R})$. Next consider

$$x^2(x-1)v = 0 \quad \text{ in } \mathcal{D}'(\mathbb{R}). \tag{2}$$

If v is a solution to (2), then v is supported in $\{0,1\}$. By a theorem from lectures it follows that v must be of the form

$$\sum_{j=0}^{J} c_j \delta_0^{(j)} + \sum_{k=0}^{K} d_k \delta_1^{(k)},$$

where J, $K \in \mathbb{N}_0$ and c_j, $d_k \in \mathbb{C}$ are constants. Now $x^2(x-1)\delta_0^{(j)} = 0$ iff $j \in \{0,1\}$ and $x^2(x-1)\delta_1^{(k)} = 0$ iff $k = 0$, so we must have $v = c_0\delta_0 + c_1\delta_0' + d_0\delta_1$ for constants c_0, c_1, $d_0 \in \mathbb{C}$. Conversely we check that any distribution of this form is a solution to (2), so that these distributions constitute the GS to (2). It follows that GS is

$$-\mathrm{pv}\big(\tfrac{1}{x}\big) - \mathrm{fp}\big(\tfrac{1}{x^2}\big) + \mathrm{pv}\big(\tfrac{1}{x-1}\big) + c_0\delta_0 + c_1\delta_0' + d_0\delta_1$$

where c_0, c_1, $d_0 \in \mathbb{C}$ are constants. [**1+3+2 marks. New Ex**]