Model solutions and marking scheme for B4.4. **March 2020**

Question 1: (a) When $f \in \mathrm{L}^1(\mathbb{R})$ its Fourier transform is

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

(i): **The Riemann-Lebesgue lemma:** If $f \in \mathrm{L}^1(\mathbb{R})$, then its Fourier transform $\widehat{f} \in \mathrm{C}_0(\mathbb{R})$.

Proof. It follows immediately from Lebesgue's dominated convergence theorem that $\widehat{f}$ is continuous. It is also clear that $\sup_{\mathbb{R}} |\widehat{f}| \leq \|f\|_1$. We prove that $\widehat{f}(\xi) \to 0$ as $\xi \to \pm\infty$ in three steps.

Step 1. If $a < b$, then $\widehat{\mathbf{1}_{(a,b)}}(\xi) = \frac{\mathrm{e}^{-ib\xi} - \mathrm{e}^{-ia\xi}}{-\mathrm{i}\xi}$ when $\xi \neq 0$ and $b - a$ for $\xi = 0$. It follows that $\widehat{\mathbf{1}_{(a,b)}} \in \mathrm{C}_0(\mathbb{R})$.

Step 2. If $s\colon \mathbb{R} \to \mathbb{C}$ is a step function, so (after changing the value of s at at most fintely many points),

$$s \in \mathrm{span}\left\{ \mathbf{1}_{(a,b)} \,:\, a,\, b \in \mathbb{R} \text{ and } a < b \right\},$$

then by linearity of $\mathcal{F}$ and Step 1 it follows that $\widehat{s} \in \mathrm{C}_0(\mathbb{R})$.

Step 3. Let $f \in \mathrm{L}^1(\mathbb{R})$. Fix $\varepsilon > 0$. Since step functions are dense in $\mathrm{L}^1(\mathbb{R})$ we find a step function s so $\|f - s\|_1 < \varepsilon/2$. Now $\sup_{\mathbb{R}} |\widehat{f} - \widehat{s}| = \sup_{\mathbb{R}} |\widehat{f - s}| \leq \|f - s\|_1 < \varepsilon/2$, so invoking Step 2 and taking $r > 0$ with $|\widehat{s}(\xi)| < \varepsilon/2$ for $|\xi| > r$ we get for $|\xi| > r$ by the traingle inequality:

$$|\widehat{f}(\xi)| \leq |\widehat{f}(\xi) - \widehat{s}(\xi)| + |\widehat{s}(\xi)| < \varepsilon$$

and we are done.

(ii): **Fourier inversion formula in** $\mathrm{L}^1(\mathbb{R})$**:** Let $f \in \mathrm{L}^1(\mathbb{R})$. Then

$$f(x) = \lim_{t \searrow 0} \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{\mathrm{i}x\xi - \frac{1}{2}(t\xi)^2}\,\mathrm{d}\xi \quad \text{in } \mathrm{L}^1(\mathbb{R}).$$

A subsequence $t_j \searrow 0$ will also converge pointwise almost everywhere. If also $\widehat{f} \in \mathrm{L}^1(\mathbb{R})$, then the formula simplifies to

$$f(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi \quad a.e.$$

Since the right-hand side is in $\mathrm{C}_0(\mathbb{R})$ by the Riemann-Lebesgue lemma it follows that f has a represnetative in $\mathrm{C}_0(\mathbb{R})$.

(iii): **Plancherel's theorem on** $\mathrm{L}^2(\mathbb{R})$**:** $\mathcal{F}\colon \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ is a linear bijection and $\|\widehat{f}\|_2 = \sqrt{2\pi}\|f\|_2$ for all $f \in \mathrm{L}^2(\mathbb{R})$. We have

$$\widehat{f}(\xi) = \lim_{j\to\infty} \int_{-j}^{j} f(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x \ \text{ and } \ f(x) = \lim_{j\to\infty} \tfrac{1}{2\pi} \int_{-j}^{j} \widehat{f}(\xi)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi \ \text{ in } \ \mathrm{L}^2(\mathbb{R}).$$

We record that in symbols: $\mathcal{F}^{-1} = \frac{1}{2\pi}\widetilde{\mathcal{F}}$ on $\mathrm{L}^2(\mathbb{R})$ and so $\mathcal{F}^2 = 2\pi\widetilde{(\cdot)}$ on $\mathrm{L}^2(\mathbb{R})$.

[1+3+3+3 marks] (All bookwork)

(b) (i): $\widehat{f_j} = \mathbf{1}_{(-j,j)}\widehat{f} \in \mathrm{L}^1 \cap \mathrm{L}^2$, the former by Cauchy-Schwarz, the latter by Plancherel and $\widehat{f_j} \to \widehat{f}$ in L^2, so by Plancherel again, $\|f_j - f\|_2 = \frac{1}{\sqrt{2\pi}}\|\widehat{f_j - f}\|_2 = \frac{1}{\sqrt{2\pi}}\|\widehat{f_j} - \widehat{f}\|_2 \to 0$.

(ii): Define f_j, g_j as in (i). Then $\widehat{f_j}$, $\widehat{g_j} \in \mathrm{L}^1 \cap \mathrm{L}^2$ and by Fubini:

$$\begin{aligned}
\widehat{\widehat{f_j} * \widehat{g_j}}(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f_j}(\xi-\eta)\widehat{g_j}(\eta)\,\mathrm{d}\eta\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f_j}(\xi-\eta)\mathrm{e}^{-\mathrm{i}x(\xi-\eta)}\widehat{g_j}(\eta)\mathrm{e}^{-\mathrm{i}x\eta}\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \widehat{\widehat{f}}_j(x)\widehat{\widehat{g}}_j(x) = (2\pi)^2 f_j(-x)g_j(-x),
\end{aligned}$$

where we used the Fourier inversion formula (both in L^1 and in L^2 will do). Since $\widehat{f_j} * \widehat{g_j} \in \mathrm{L}^1$ we get from Fourier inversion formula on L^1:

$$\begin{aligned}
\big(\widehat{f_j} * \widehat{g_j}\big)(\xi) &= \tfrac{1}{2\pi}\lim_{t\searrow 0}\int_{-\infty}^{\infty} \widehat{\widehat{f_j} * \widehat{g_j}}(x)\mathrm{e}^{\mathrm{i}x\xi - \frac{1}{2}(tx)^2}\,\mathrm{d}x \\
&= 2\pi \lim_{t\searrow 0}\int_{-\infty}^{\infty} f_j(-x)g_j(-x)\mathrm{e}^{\mathrm{i}x\xi - \frac{1}{2}(tx)^2}\,\mathrm{d}x \\
(\widetilde{f_j}\widetilde{g_j} \in \mathrm{L}^1)\ &= 2\pi \int_{-\infty}^{\infty} f_j(-x)g_j(-x)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}x \\
(-x \mapsto x)\ &= 2\pi \int_{-\infty}^{\infty} f_j(x)g_j(x)\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x = 2\pi\widehat{(f_j g_j)}(\xi).
\end{aligned}$$

Since, using triangle and Cauchy-Schwarz inequalities, $\|f_j g_j - fg\|_1 \leq \|(f_j - f)g_j\|_1 + \|f(g_j - g)\|_1 \leq \|f_j - f\|_2\|g_j\|_2 + \|f\|_2\|g_j - g\|_2 \to 0$ we get

$$\|\mathcal{F}(f_j g_j) - \mathcal{F}(fg)\|_\infty \leq \|f_j g_j - fg\|_1 \to 0.$$

By the definitions

$$\big(\widehat{f_j} * \widehat{g_j}\big)(\xi) = \int_{(-j,j)\cap(\xi-j,\xi+j)} \widehat{f}(\xi-\eta)\widehat{g}(\eta)\,\mathrm{d}\eta$$

and since $\eta \mapsto \widehat{f}(\xi - \eta)\widehat{g}(\eta)$ is integrable by Cauchy-Schwarz we deduce by Lebesgue's dominated convergence theorem $\widehat{f_j} * \widehat{g_j} \to \widehat{f} * \widehat{g}$ pointwise on $\mathbb{R}$.

[3+5 marks] (New variant of bookwork)

(c) *Proof.* '$\subseteq$' Let $f \in \mathrm{L}^1$. Then $g_1 := \sqrt{|f|}$, $g_2 := \sqrt{|f|}\mathrm{sgn}(f) \in \mathrm{L}^2$ so by (b)(ii)

$$\widehat{f} = \widehat{g_1 g_2} = \tfrac{1}{2\pi}\widehat{g_1} * \widehat{g_2} = \widehat{\tfrac{g_1}{2\pi}} * \widehat{g_2}$$

and Plancherel guarantees that $\widehat{\frac{g_1}{2\pi}}$, $\widehat{g_2} \in \mathrm{L}^2$.

'$\supseteq$' Let f, $g \in \mathrm{L}^2$. Then by Plancherel and Fourier inversion we have $f = \frac{1}{2\pi}\widehat{f_1}$ with $f_1 := \widetilde{\widehat{f}} \in \mathrm{L}^2$ and similarly for g, hence by (b)(ii),

$$f * g = \tfrac{1}{(2\pi)^2}\widehat{f_1} * \widehat{g_1} = \tfrac{1}{2\pi}\widehat{f_1 g_1} = \widehat{\tfrac{f_1}{2\pi} g_1}$$

and this is the required identity since $\frac{f_1}{2\pi}g_1 \in \mathrm{L}^1$ by Cauchy-Schwarz.

[7 marks] (New example)

Question 2: (a) **Schwartz class of test functions on $\mathbb{R}^n$:**

$$\mathscr{S}(\mathbb{R}^n) := \left\{ \varphi \in \mathrm{C}^\infty(\mathbb{R}^n) : \, S_{\alpha,\beta}(\varphi) < \infty \text{ for all } \alpha,\, \beta \in \mathbb{N}_0^n \right\},$$

where $S_{\alpha,\beta}(\varphi) := \sup_{x\in\mathbb{R}^n}\left|x^\alpha(\partial^\beta\varphi)(x)\right|$. $\phi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$ if $S_{\alpha,\beta}(\phi_j) \to 0$ for all α, $\beta \in \mathbb{N}_0^n$.

Tempered distribution on $\mathbb{R}^n$: $u \colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ linear and $\langle u, \phi_j\rangle \to 0$ when $\phi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$.

$u_j \to 0$ in $\mathscr{S}'(\mathbb{R}^n)$ if $\langle u_j, \phi\rangle \to 0$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

[1+1 marks] (All bookwork)

(i): **Fourier bounds:** For k, $l \in \mathbb{N}_0$ put $\overline{S}_{k,l}(\phi) := \max_{|\alpha|\le k, |\beta|\le l} S_{\alpha,\beta}(\phi)$. Then there exist constants $c = c(n,k,l) \ge 0$ so $\overline{S}_{k,l}(\widehat{\phi}) \le c\overline{S}_{l+n+1,k}(\phi)$ holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Fourier inversion formula on $\mathscr{S}(\mathbb{R}^n)$: $\mathcal{F} \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a linear bijection with

$$\mathcal{F}^{-1}(\phi)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\phi(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi.$$

(Or: $\mathcal{F}^{-1} = (2\pi)^{-n}\widetilde{\mathcal{F}}$.)

[2+2 marks] (All bookwork)

(ii): If $u \in \mathscr{S}'(\mathbb{R}^n)$, then $\widehat{u} \in \mathscr{S}'(\mathbb{R}^n)$ defined by rule

$$\langle \widehat{u}, \phi\rangle := \langle u, \widehat{\phi}\rangle, \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

(Not required: well-defined since clearly defined and linear on $\mathscr{S}(\mathbb{R}^n)$, and if $\phi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$, then by Fourier bounds $\widehat{\phi}_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$ too, hence $\langle \widehat{u}, \phi_j \rangle \to 0$.) Consistent: by the product rule

$$\int_{\mathbb{R}^n} \widehat{\phi}\psi \,\mathrm{d}x = \int_{\mathbb{R}^n} \phi\widehat{\psi} \,\mathrm{d}x \quad \forall\, \phi,\, \psi \in \mathscr{S}(\mathbb{R}^n)$$

that is a straight forward consequence of Fubini's theorem.

[1+1 marks] (All Bookwork)

(iii): **Fourier inversion formula in** $\mathscr{S}'(\mathbb{R}^n)$**:** $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is a linear bijection with $\mathcal{F}^{-1} = (2\pi)^{-n}\widetilde{\mathcal{F}}$, where $\widetilde{\mathcal{F}}u = \widetilde{\widehat{u}} = \widehat{\widetilde{u}}$ and $\langle \widetilde{u}\, \phi \rangle := \langle u, \widetilde{\phi} \rangle$, $\widetilde{\phi}(x) := \phi(-x)$.

Proof. Clear that $\mathcal{F}\colon \mathscr{S}' \to \mathscr{S}'$ is linear. We first check that $\widetilde{\widehat{u}} = \widehat{\widetilde{u}}$: for $\phi \in \mathscr{S}$ the definitions give $\langle \widehat{\widetilde{u}}, \phi \rangle = \langle u, \widetilde{\widehat{\phi}} \rangle$ and for $\xi \in \mathbb{R}^n$ we calculate

$$\widetilde{\widehat{\phi}}(\xi) = \widehat{\phi}(-\xi) = \int_{\mathbb{R}^n} \phi(x)\mathrm{e}^{\mathrm{i}\xi\cdot x}\,\mathrm{d}x \overset{y=-x}{=} \int_{\mathbb{R}^n} \phi(-y)\mathrm{e}^{-\mathrm{i}\xi\cdot y}\,\mathrm{d}y = \widehat{\widetilde{\phi}}(\xi),$$

consequently $\langle \widehat{\widetilde{u}}, \phi \rangle = \langle u, \widetilde{\widehat{\phi}} \rangle = \langle u, \widehat{\widetilde{\phi}} \rangle = \langle \widetilde{\widehat{u}}, \phi \rangle$.

We next check that $(2\pi)^{-n}\widetilde{\mathcal{F}}\mathcal{F} = \mathrm{Id} = \mathcal{F}\big((2\pi)^{-n}\widetilde{\mathcal{F}}\big)$. For $\phi \in \mathscr{S}$ we get by use of definitions and the Fourier inversion formula in $\mathscr{S}$:

$$\begin{aligned}\big\langle (2\pi)^{-n}\widetilde{\mathcal{F}}\mathcal{F}u, \phi \big\rangle =& \big\langle u, \mathcal{F}\big((2\pi)^{-n}\widetilde{\mathcal{F}}\phi\big)\big\rangle \\ =& \langle u, \phi \rangle = \big\langle u, (2\pi)^{-n}\widetilde{\mathcal{F}}\big(\mathcal{F}\phi\big)\big\rangle \\ =& \big\langle \mathcal{F}\big((2\pi)^{-n}\widetilde{\mathcal{F}}u\big), \phi \big\rangle,\end{aligned}$$

and we are done.

[1+3 marks] (All bookwork)

(b): $\delta_0 \in \mathscr{S}'(\mathbb{R})$ so we have for $\phi \in \mathscr{S}(\mathbb{R})$:

$$\langle \widehat{\delta_0}, \phi \rangle = \langle \delta_0, \widehat{\phi} \rangle = \widehat{\phi}(0) = \int_{\mathbb{R}} \phi \,\mathrm{d}x,$$

so $\widehat{\delta_0} = 1$. Since

$$\langle \mathbf{1}_{(-j,j)}, \phi \rangle = \int_{-j}^{j} \phi \,\mathrm{d}x \to \int_{\mathbb{R}} \phi \,\mathrm{d}x$$

and $\mathcal{F}$ is $\mathscr{S}'$ continuous (immediate from definitions) the Fourier inversion formula on $\mathscr{S}'$ yields

$$\delta_0 = (2\pi)^{-1}\widetilde{\widehat{1}} = \lim_{j\to\infty} (2\pi)^{-1}\widetilde{\widehat{\mathbf{1}_{(-j,j)}}} = \lim_{j\to\infty} \tfrac{1}{2\pi} \int_{-j}^{j} \mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}x.$$

Since $\delta_0 = \widetilde{\delta_0}$ the conclusion follows from the $\mathscr{S}'$ continuity of the operation $\widetilde{(\cdot)}$.
Write $\sin^2 x = \frac{1}{2} - \frac{1}{4}\mathrm{e}^{\mathrm{i}2x} - \frac{1}{4}\mathrm{e}^{-\mathrm{i}2x}$ and so by the above and the translation rule

$$\mathcal{F}_{x\to\xi}\big(\sin^2 x\big) = \pi\delta_0 - \tfrac{\pi}{2}\delta_2 - \tfrac{\pi}{2}\delta_{-2}.$$

[3+3 marks] (Seen before and new example)
(c): Let $\phi \in \mathscr{S}(\mathbb{R})$ and estimate using the Fourier bounds:

$$\begin{aligned}\big|c_k\langle \mathrm{e}^{\mathrm{i}kx},\phi\rangle\big| = \big|c_k\widehat{\phi}(-k)\big| &\leq M\big(1+k^m\big)\big|\widehat{\phi}(-k)\big| \\ &= \tfrac{M}{1+k^2}\big(1+k^2+|-k|^m+|-k|^{m+2}\big)\big|\widehat{\phi}(-k)\big| \\ &\leq \tfrac{4M\overline{S}_{m+2,0}(\widehat{\phi})}{1+k^2} \\ &\leq \tfrac{4Mc}{1+k^2}\overline{S}_{2,m+2}(\phi)\end{aligned}$$

for all $k \in \mathbb{N}_0$. It follows that the series $\sum_{k=0}^{\infty} c_k\langle \mathrm{e}^{\mathrm{i}kx},\phi\rangle$ is absolutely convergent and that

$$\big|\langle u,\phi\rangle\big| \leq \left(\sum_{k=0}^{\infty}\tfrac{4Mc}{1+k^2}\right)\overline{S}_{2,m+2}(\phi)$$

for all $\phi \in \mathscr{S}(\mathbb{R})$. Thus $u \in \mathscr{S}'(\mathbb{R})$. If $r \in (0,1)$, then the function $F(r\mathrm{e}^{\mathrm{i}x})$ is a bounded continuous function of $x \in \mathbb{R}$, and if $\phi \in \mathscr{S}(\mathbb{R})$, then from above bound we get

$$\left|\sum_{k=0}^{\infty}\big(c_k\widehat{\phi}(-k) - c_k r^k\widehat{\phi}(-k)\big)\right| \leq \left(\sum_{k=0}^{\infty}\tfrac{4Mc}{1+k^2}\right)\overline{S}_{2,m+2}(\phi)(1-r) \to 0$$

as $r \nearrow 1$. Consequently, $F(r\mathrm{e}^{\mathrm{i}x}) \to u$ in $\mathscr{S}'(\mathbb{R})$ as $r \nearrow 1$.
Note that $F'(z) = \sum_{k=1}^{\infty} kc_k z^{k-1}$ and $u' = \sum_{k=1}^{\infty} c_k\mathrm{i}k\mathrm{e}^{\mathrm{i}kx}$ by $\mathscr{S}'$ continuity of differentiation, consequently as $r \nearrow 1$,

$$F'(r\mathrm{e}^{\mathrm{i}x}) \to -\mathrm{i}\mathrm{e}^{-\mathrm{i}x}u' \ \text{ in } \ \mathscr{S}'(\mathbb{R}).$$

[4+3 marks] (New examples)

Question 3: (a)(i): We estimate for $l \in \{0,1,2\}$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$:

$$\begin{aligned}\big|f^{(l)}(x+2\pi k)\big| &= \tfrac{1+|x+2\pi k|^2}{1+|x+2\pi k|^2}\big|f^{(l)}(x+2\pi k)\big| \\ &\leq \tfrac{2\overline{S}_{2,l}(f)}{1+|x+2\pi k|^2}.\end{aligned}$$

It follows from this with $l = 0$ that the series defining $Pf(x)$ is absolutely convergent and so defines a 2π-periodic function $Pf\colon \mathbb{R} \to \mathbb{C}$. If $|x| \leq 2\pi$ then the above bound gives

$$\left|f^{(l)}(x+2\pi k)\right| \leq \tfrac{2\overline{S}_{2,l}(f)}{1+|x+2\pi k|^2} \leq \tfrac{2\overline{S}_{2,l}(f)}{1+(2\pi)^2(k^2-1)}$$

for all $|k| \geq 2$, and consequently by Weierstrass' M-test the series $\sum_{k\in\mathbb{Z}} f^{(l)}(x+2\pi k)$ is uniformly convergent in $x \in [-2\pi, 2\pi]$. The function Pf is therefore C^2 on $(-2\pi, 2\pi)$, and therefore by 2π-periodicity a C^2 function on $\mathbb{R}$.

[4 marks] (All bookwork)

(ii): By (i) and hint we have $(Pf)(x) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx}$ uniformly in $x \in \mathbb{R}$. Now for each $k \in \mathbb{Z}$ we get by the uniform convergence of the series defining Pf:

$$\begin{aligned} c_k = \tfrac{1}{2\pi}\int_0^{2\pi} Pf(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x &= \tfrac{1}{2\pi}\sum_{j\in\mathbb{Z}}\int_0^{2\pi} f(x+2\pi j)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \tfrac{1}{2\pi}\sum_{j\in\mathbb{Z}}\int_{2\pi j}^{2\pi(j+1)} f(y)\mathrm{e}^{-\mathrm{i}k(y-2\pi j)}\,\mathrm{d}y \\ &= \tfrac{1}{2\pi}\sum_{j\in\mathbb{Z}}\int_{2\pi j}^{2\pi(j+1)} f(y)\mathrm{e}^{-\mathrm{i}ky}\,\mathrm{d}y \\ &= \tfrac{\widehat{f}(k)}{2\pi} \end{aligned}$$

as required. **[3 marks]** (Variant of bookwork)

(iii): Since $t > 0$ the function $x \mapsto \mathrm{e}^{-t|x|}$ is integrable on $\mathbb{R}$ and its Fourier transform is

$$\begin{aligned} \mathcal{F}_{x\to\xi}\big(\mathrm{e}^{-t|x|}\big) &= \int_{-\infty}^0 \mathrm{e}^{(t-\mathrm{i}\xi)x}\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-(t+\mathrm{i}\xi)x}\,\mathrm{d}x \\ &= \tfrac{1}{t-\mathrm{i}\xi} + \tfrac{1}{t+\mathrm{i}\xi} = \tfrac{2t}{t^2+\xi^2}. \end{aligned}$$

Since the function $f(x) := \frac{2t}{t^2+x^2}$ for each fixed $t > 0$ is C^2 and $\overline{S}_{2,2}(f) < \infty$ we can apply (ii). By Fourier inversion in $\mathscr{S}$: $\widehat{f}(\xi) = \mathcal{F}^2(\mathrm{e}^{-t|\cdot|}) = 2\pi\mathrm{e}^{-t|\xi|}$ and then by (ii) follows

$$\sum_{k\in\mathbb{Z}} \tfrac{2t}{t^2+(x+2\pi k)^2} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-t|k|+\mathrm{i}kx}$$

for all $x \in \mathbb{R}$ and $t > 0$. Take $x = 0$ and $t = 2\pi$ to conclude. **[5 marks]** (New example) (b): **Plancherel theorem for Fourier series:** If $f\colon \mathbb{R} \to \mathbb{C}$ is a 2π-periodic $\mathrm{L}^2_{\mathrm{loc}}$ function, then

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx}, \quad c_k = \tfrac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,$$

holds in $\mathrm{L}^2(0,2\pi]$. Furthermore, Parseval's formula holds:

$$\tfrac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,\mathrm{d}x = \sum_{k\in\mathbb{Z}} |c_k|^2.$$

Conversely, if $(b_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then the series $\sum_{k\in\mathbb{Z}} b_k \mathrm{e}^{\mathrm{i}kx}$ converges in $\mathrm{L}^2(0,2\pi]$ to a 2π-periodic $\mathrm{L}^2_{\mathrm{loc}}$ function. **[2 marks]** (All bookwork)

(i): Put $f_\varepsilon := \rho_\varepsilon * f$, where $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}$. Then f_ε is a 2π-periodic C^∞ function and $\|f - f_\varepsilon\|_{\mathrm{L}^2(0,2\pi)} + \|f' - f'_\varepsilon\|_{\mathrm{L}^2(0,2\pi)} \to 0$ as $\varepsilon \searrow 0$. If $c_k(g)$ denote the Fourier coefficients for the function g, then we get by partial integration:

$$c_k(f'_\varepsilon) = \tfrac{1}{2\pi}\int_0^{2\pi} f'_\varepsilon(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = ikc_k(f_\varepsilon)$$

Since $c_k(f'_\varepsilon) \to c_k(f')$ and $c_k(f_\varepsilon) \to c_k$ as $\varepsilon \searrow 0$ we find $c_k(f') = \mathrm{i}kc_k$ as required. **[2 marks]** (Seen before)

By Parseval's formula

$$\begin{aligned}\int_0^{2\pi} |f(x) - c_0|^2\,\mathrm{d}x = 2\pi\sum_{k\neq 0}|c_k|^2 &= 2\pi\sum_{k\neq 0}\tfrac{1}{k^2}|\mathrm{i}kc_k|^2 \\ &\leq 2\pi\sum_{k\neq 0}|\mathrm{i}kc_k|^2 \\ &= 2\pi\int_0^{2\pi}|f'(x)|^2\,\mathrm{d}x.\end{aligned}$$

The equality holds precisely when $f = c_0 + c_{-1}\mathrm{e}^{-\mathrm{i}x} + c_1\mathrm{e}^{\mathrm{i}x}$.

[3+1 marks] (Seen before)

(ii): Extend g to odd 2π-periodic function (still denoted) $g\colon \mathbb{R} \to \mathbb{C}$. Clearly $g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$. It is clear that g is C^1 away from $\pi\mathbb{Z}$ and since $g(0) = g(\pi) = 0$ the function g is continuous so by integration by parts we see that the distributional derivative g' is represented by the usual derivative (on $\mathbb{R}\setminus\pi\mathbb{Z}$) and hence that it in particular is in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$. **[2 marks]**

By (i) we have

$$\int_0^{2\pi} |g(x) - c_0(g)|^2 \, \mathrm{d}x \leq \int_0^{2\pi} |g'(x)|^2 \, \mathrm{d}x$$

with equality exactly when $g = c_0 + c_{-1}\mathrm{e}^{-\mathrm{i}x} + c_1\mathrm{e}^{\mathrm{i}x}$. Because g is odd and 2π-periodic, $c_0(g) = 0$ and

$$\int_0^{2\pi} |g(x) - c_0(g)|^2 \, \mathrm{d}x = 2\int_0^{\pi} |g(x)|^2 \, \mathrm{d}x,$$

g' is even and 2π-periodic so

$$\int_0^{2\pi} |g'(x)|^2 \, \mathrm{d}x = 2\int_0^{\pi} |g'(x)|^2 \, \mathrm{d}x.$$

It follows that

$$\int_0^{\pi} |g(x)|^2 \, \mathrm{d}x \leq \int_0^{\pi} |g'(x)|^2 \, \mathrm{d}x.$$

Equality holds precisely when $g = c_0 + c_{-1}\mathrm{e}^{-\mathrm{i}x} + c_1\mathrm{e}^{\mathrm{i}x}$ that for odd 2π-periodic functions require $c_0 = 0$ and $c_{-1} = -c_1$, hence equality holds precisely when $g = c \sin$ for some $c \in \mathbb{C}$.

[3+2 marks] (New example)