1. (a) (i)

emission $\gamma\sigma T^4$ per unit area.

$$4\pi R^2 C \frac{d\bar{T}}{dt} = \pi R^2 Q(1-a) - 4\pi R^2 \sigma\gamma \bar{T}^4$$

rate of change of energy in atmosphere; radiation from sun sees area πR^2; Stefan-Boltzmann radiation from Earth's surface

a = albedo (fraction of shortwave radiation reflected)

γ = greenhouse factor (fraction of longwave emission that make it out to space).

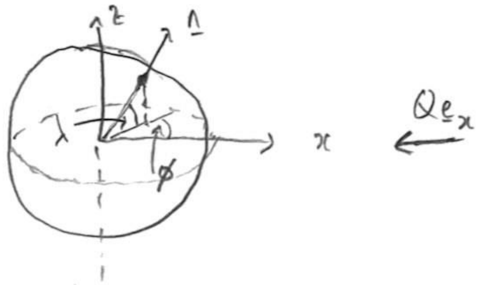
$\therefore$ divide by $\pi R^2 \Rightarrow$ $\boxed{4C\frac{d\bar{T}}{dt} = Q(1-a) - 4\sigma\gamma\bar{T}^4}$ [3]

(ii) equilibrium temp $\bar{T}_0 = \left(\dfrac{Q(1-a)}{4\sigma\gamma}\right)^{1/4}$

- In glacial periods the planetary albedo is larger so $\bar{T}$ smaller.
- Increase greenhouse gas emissions cause greenhouse factor γ to decrease $\Rightarrow$ larger $\bar{T}$ [3]

[standard]

(b) (i)



From the diagram, the normal is $\boxed{(\cos\lambda\cos\phi, \cos\lambda\sin\phi, \sin\lambda)}$ [2]

The solar radiation per unit area perpendicular to the normal is the dot product $Q\underline{e}_x\cdot\underline{n} = Q\cos\lambda\cos\phi$ but of course only if it is facing the sun, so only if $-\frac{\pi}{2}<\phi<\frac{\pi}{2}$, ie. $\cos\phi>0$., so we need $Q\cos\lambda[\cos\phi]_+$.

For each element of surface area $(R^2\cos\lambda\, d\lambda\, d\phi)$, the local energy balance is

$$\boxed{C\frac{dT}{dt} = \underbrace{Q(1-a)\cos\lambda(\cos\phi)_+}_{\text{solar radiation per unit area, factoring in albedo.}} - \underbrace{\sigma\gamma T^4}_{\text{longwave emission as in (a)}}} \quad (*)$$

[3]

(ii) If $T=\bar{T}$ is independent of λ and ϕ, we integrate.

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} C\frac{d\bar{T}}{dt}\cos\lambda\, d\lambda\, d\phi = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} Q(1-a)\cos\lambda(\cos\phi)_+\cos\lambda\, d\lambda\, d\phi - \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\sigma\gamma\bar{T}^4\cos\lambda\, d\lambda\, d\phi$$

Note $\int_{-\pi/2}^{\pi/2}\cos\lambda\, d\lambda = 2$, $\int_{-\pi/2}^{\pi/2}\cos^2\lambda\, d\lambda = \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1+\cos 2\lambda\, d\lambda = \frac{\pi}{2}$, $\int_0^{2\pi} d\phi = 2\pi$,

and $\int_0^{2\pi}(\cos\phi)_+\, d\phi = \int_{-\pi/2}^{\pi/2}\cos\phi\, d\phi = 2$.

so this gives $4\cancel{\pi} C\frac{d\bar{T}}{dt} = Q(1-a)\cancel{\pi} - 4\cancel{\pi}\sigma\gamma\bar{T}^4$, which is as in (a). [4]

[new, with guidance]

(c) (i) $t = \frac{\hat{t}}{\omega}$ $\quad T = \bar{T}_0 \hat{T}$ $\quad$ where $\bar{T}_0^4 = \frac{Q(1-\alpha)}{4\sigma\gamma}$ given, in (*). with $\phi = \omega t$.

$$\hat{\omega} \frac{d\hat{T}}{d\hat{t}} = \cos\lambda (\cos\hat{t})_+ - \frac{1}{4}\hat{T}^4$$

where $\hat{\omega} = \frac{\omega C \bar{T}_0}{Q(1-\alpha)}$. The timescale $\frac{C\bar{T}_0}{Q(1-\alpha)}$ for radiative adjustment is on the order of a month (bookwork), and $\omega = \frac{1}{1\,\text{day}}$, so $\hat{\omega}$ is relatively large. [2]

Hence $\hat{T}$ is roughly constant and we can write $\hat{T} = \hat{T}_0(\lambda) + \frac{1}{\hat{\omega}}\hat{T}_1(\lambda, t)$.

(ii) $$\frac{d\hat{T}_1}{d\hat{t}} = \cos\lambda(\cos\hat{t})_+ - \frac{1}{4}\hat{T}_0^4\left(1 + \frac{1}{\hat{\omega}}\frac{\hat{T}_1}{\hat{T}_0}\right)^4$$

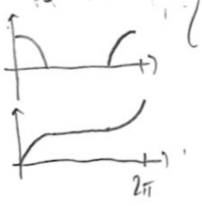
$$\boxed{\frac{d\hat{T}_1}{d\hat{t}} \approx \cos\lambda(\cos\hat{t})_+ - \frac{1}{4}\hat{T}_0^4}$$ to leading order.

averaging in time $\left(\int_0^{2\pi} \cdot \, d\hat{t}\right)$, noting $\int_0^{2\pi} (\cos\hat{t})_+ d\hat{t} = 2$ as before.

$$\Rightarrow \quad 0 = \cos\lambda \cdot 2 - \frac{1}{4}\hat{T}_0^4 \cdot 2\pi \quad \Rightarrow \quad \boxed{\hat{T}_0 = \left(\frac{4}{\pi}\cos\lambda\right)^{1/4}}$$ [3]

Subtracting this time average from the equation given $\frac{d\hat{T}_1}{d\hat{t}} = \cos\lambda\left[(\cos\hat{t})_+ - \frac{1}{\pi}\right]$

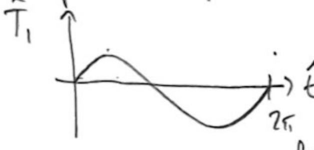
$$S(t) = \int_0^t (\cos\hat{t})_+ d\hat{t} = \begin{cases} \sin t & 0 < t \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < t \le \frac{3\pi}{2} \\ 2 + \sin t & \frac{3\pi}{2} < t \le 2\pi \end{cases}$$



so $\hat{T}_1 = \cos\lambda\left[S(\hat{t}) - \frac{\hat{t}}{\pi} + c\right]$

constant to make average to zero – in fact $c = 0$ to do this.

$$\boxed{\hat{T}_1 = \cos\lambda\left[S(\hat{t}) - \frac{\hat{t}}{\pi}\right]}$$



[3]

The temperature is maximum when $\cos\hat{t} = \frac{1}{\pi}$, so $\hat{t} = \cos^{-1}\left(\frac{1}{\pi}\right) \approx 1.25$

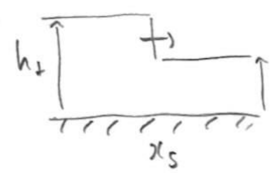
which is around 4.8 hours after noon ($\hat{t} = 0$)

[2]

[new].

2.

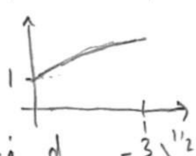
(a) (i) If $\delta \ll 1$ and $\delta F^2 \ll 1$, momentum equation gives approximately $u^2 = h$, so $\boxed{u = h^{1/2}}$
and hence $q = hu = h^{3/2}$, so $\boxed{\frac{\partial h}{\partial t} + \frac{3}{2} h^{1/2} \frac{\partial h}{\partial x} = E}$ [2]

(ii) 

In the frame of the shock, the flux passing from upstream must equal the flux continuing downstream (mass conservation!)

so $q_+ - \dot{x}_s h_+ = q_- - \dot{x}_s h_-$ $\Rightarrow$ $\dot{x}_s = \frac{q_+ - q_-}{h_+ - h_-} = \frac{h_+^{3/2} - h_-^{3/2}}{h_+ - h_-}$ [2]

If $h_- = \lambda h_+$, then is $\dot{x}_s = \underbrace{h_+^{1/2}}_{\text{speed upstream}} \underbrace{\left(\frac{1 - \lambda^{3/2}}{1 - \lambda} \right)}_{\text{this factor is } \geq 1}$



$1 - \frac{1}{2}\lambda^{1/2}(3 - \lambda) \geq 0$

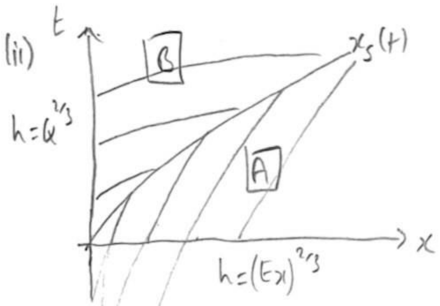
since $\frac{d}{d\lambda} = \frac{-\frac{3}{2}\lambda^{1/2}}{1 - \lambda} + \frac{(1 - \lambda^{3/2})}{(1 - \lambda)^2} = \frac{1 - \frac{3}{2}\lambda^{1/2} + \frac{1}{2}\lambda^{3/2}}{(1 - \lambda)^2} \geq 0$

[3].

[standard].

(b) (i) The steady state river clearly has $q = h^{3/2} = Ex$, so $h = (Ex)^{2/3}$ at $t = 0$.

If the flux is Q at $x = 0$, it gives $h^{3/2} = Q$ $\Rightarrow$ $h = Q^{2/3}$ at $x = 0$. [2]

(ii) 

- characteristics have $\dot{t} = 1$ $\dot{x} = \frac{3}{2} h^{1/2}$ $\dot{h} = E$.
- For characteristics coming from $t = 0$, take $x = x_0$, $h = (Ex_0)^{2/3}$ at $t = 0$.

then $\frac{dh}{dx} = \frac{\dot{h}}{\dot{x}} = \frac{E}{\frac{3}{2}h^{1/2}}$ $\Rightarrow$ $h^{3/2} = Ex$ $\Rightarrow$ $\boxed{h = (Ex)^{2/3}}$ on all such characteristics

region $\boxed{A}$

The characteristics have $\dot{x} = \frac{3}{2} h^{1/2} = \frac{3}{2} (Ex)^{1/3}$
$\Rightarrow$ $x^{2/3} = E^{1/3} t + x_0^{2/3}$, [3]

- For characteristics coming from $x = 0$, take $x = 0$, $h = Q^{2/3}$ at $t = t_0$.

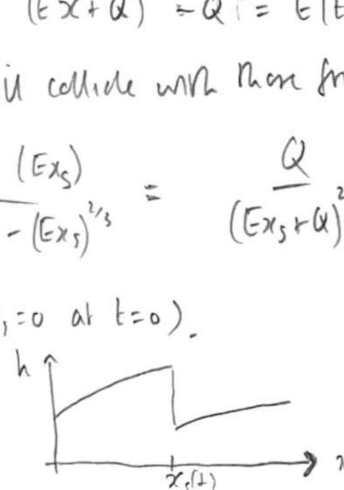
then $\frac{dh}{dx} = \frac{E}{\frac{3}{2}h^{1/2}}$ $\Rightarrow$ $h^{3/2} = Ex + Q$ $\Rightarrow$ $\boxed{h = (Ex + Q)^{2/3}}$ on all such characteristics. region $\boxed{B}$

These characteristics have $\dot{x} = \frac{3}{2} h^{1/2} = \frac{3}{2} (Ex + Q)^{1/3}$ $\Rightarrow$ $(Ex + Q)^{2/3} - Q^{2/3} = E(t - t_0)$ [2].

- Note $\dot{x}$ is larger on characteristics from $x = 0$, so they will collide with those from $t = 0$ $\Rightarrow$ <u>shock</u>.

The shock at x_s has $\dot{x}_s = \frac{h_+^{3/2} - h_-^{3/2}}{h_+ - h_-} = \frac{(Ex_s + Q) - (Ex_s)}{(Ex_s + Q)^{2/3} - (Ex_s)^{2/3}} = \frac{Q}{(Ex_s + Q)^{2/3} - (Ex_s)^{2/3}}$

$\Rightarrow$ $\boxed{\frac{3}{5}\left[(Ex_s + Q)^{5/3} - (Ex_s)^{5/3} - Q^{5/3} \right] = EQt}$ ($x_s = 0$ at $t = 0$). [3]

So 

[new, but standard]

(c) (i) Conservation of contaminant is $\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(huc) = 0$

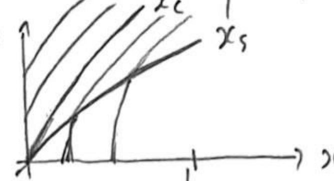
with mass conservation this gives $h\left(\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x}\right) + Ec = 0$, & since $u = h^{1/2}$ this gives

$$\boxed{\frac{\partial c}{\partial t} + h^{1/2}\frac{\partial c}{\partial x} = -\frac{Ec}{h}}$$

with $c = c_0$ at $x=0$. (since this is concentration of spillage)
& $c = 0$ at $t=0$ (since no contaminant initially. [2]

(ii) From (b) we have $h = \begin{cases} (Ex)^{2/3} & x > x_s(t) \\ (Ex+Q)^{2/3} & x < x_s(t) \end{cases}$, and characteristic eqns for c are

$\dot{t} = 1 \quad \dot{x} = h^{1/2}, \quad \dot{c} = -\frac{Ec}{h}$



Note that the characteristic speed here $h^{1/2}$ is the water speed, and we showed in (b) that the shock moves faster than the upstream water speed, so characteristics from $x=0$ must remain in $x < x_s$ all the time, with the first one (from $t=0$) being given by

$\dot{x} = (Ex+Q)^{1/3} \Rightarrow \boxed{\frac{3}{2}\left((Ex+Q)^{2/3} - Q^{2/3}\right) = Et.}$ ← labelled $x_c(t)$ above.

Thus for $x > x_c$, the characteristics from $t=0$ indicate that $c=0$ (a conservation condition must apply where they pass through the shock, but it cannot generate any c). [1]

On the characteristics from $x=0$, $\frac{dc}{dx} = \frac{\dot{c}}{\dot{x}} = -\frac{Ec}{h^{3/2}} = -\frac{Ec}{Ex+Q}$ with $c = c_0$ at $x=0$

$\Rightarrow \ln\left(\frac{c}{c_0}\right) = -\ln\left(\frac{x+\frac{Q}{E}}{Q/E}\right)$ so $\boxed{c = c_0\left(\frac{Q}{Ex+Q}\right)}$ [3]

(in fact this is just conservation $\frac{d}{dt}(h^{3/2}c) = 0$ along characteristics.)

so $\boxed{c = \begin{cases} c_0\frac{Q}{Ex+Q} & x < x_c(t) \\ 0 & x > x_c(t) \end{cases}}$

- The observer at $x=L$ will first notice the spillage when the shock reaches them (at $t = \frac{3}{5E}\left[(EL+Q)^{5/3} - (EL)^{5/3} - Q^{5/3}\right]$) but the water upstream of the shock does not contain any contaminant since it is only moved at the upstream water speed.

They notice the contaminant when the contaminant 'front' x_c reaches them. i.e. when

$$\boxed{t = \frac{3}{2E}\left((EL+Q)^{2/3} - Q^{2/3}\right)}$$ [3]

3.

(a) (i) ① $0 = -p_x + \tau_z + \rho g \sin\theta$ ② $0 = -p_z - \rho g \cos\theta$ ③ $u_z = 2A\tau^n$

② + $p=0$ at $z=s$ ⇒ $p = \rho g \cos\theta (s-z)$.

① + $\tau = 0$ at $z = s$ ⇒ $\tau = \rho g(\sin\theta - \cos\theta \frac{\partial s}{\partial x})(s-z)$

③ + $u = C\tau^m$ at $z = b$ ⇒ $u = 2A(\rho g \sin\theta)^n \left(1 - \cot\theta \frac{\partial s}{\partial x}\right)^n \left[\frac{(s-b)^{n+1} - (s-z)^{n+1}}{n+1}\right] + C(\rho g \sin\theta)^m \left(1 - \cot\theta \frac{\partial s}{\partial x}\right)^m (s-b)^m$

so $q = \int_b^s u\,dz = \boxed{2A(\rho g \sin\theta)^n \left(1 - \cot\theta \frac{\partial s}{\partial x}\right)^n \frac{H^{n+2}}{n+2} + C(\rho g \sin\theta)^m \left(1 - \cot\theta \frac{\partial s}{\partial x}\right)^m H^{m+1}}$

[4]
[standard].

(ii) Melting doesn't occur for $x < X$ because the altitude is too high ⇒ cold air temperatures. Air temperature increases with lower altitude and hence larger x, for $x > X$, eg. linear dependence reflects constant lapse rate. Colder climate ⇒ larger X.

[2]
[new].

(iii) $\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a - m$ with $m = \lambda(x - X)_+$ and q as above.

Scale $[q] = a[x] = C(\rho g \sin\theta)^m [H]^{m+1}$ and $a = \lambda[x]$, (scale $X \sim [x]$, $s, b \sim [H]$ too).

$$\Rightarrow \boxed{[x] = \frac{a}{\lambda} \qquad [H] = \left(\frac{a^2}{\lambda C(\rho g \sin\theta)^m}\right)^{\frac{1}{m+1}}}$$

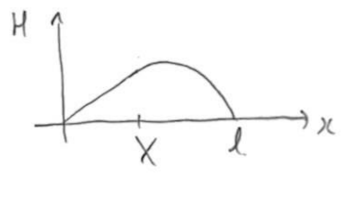
Then $\varepsilon \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\alpha\left(1 - \mu\frac{\partial s}{\partial x}\right)^n \frac{H^{n+2}}{n+2} + \left(1 - \mu \frac{\partial s}{\partial x}\right)^m H^{m+1}\right] = 1 - (x - X)_+$

where $\boxed{\varepsilon = \frac{[H]}{t_0 a}, \quad \alpha = \frac{2A(\rho g \sin\theta)^{n-m}[H]^{n-m+1}}{C}, \quad \mu = \frac{[H]}{[x]}\cot\theta}$

[4]
[standard]

(b) If $\varepsilon = \alpha = \mu = 0$, we have $\frac{d}{dx}(H^{m+1}) = 1 - (x - X)_+$ and need $H = 0$ at $x = 0$,

$$\Rightarrow H^{m+1} = x - \tfrac{1}{2}(x - X)_+^2 = \begin{cases} x & x < X \\ x(1+X) - \frac{1}{2}x^2 - \frac{1}{2}X^2 & x \geq X \end{cases}$$



$x = l$ is where $H = 0$, ie. $l^2 - 2(1+X)l + X^2 = 0$

$\Rightarrow (l - (1+X))^2 = (1+X)^2 - X^2 = 1 + 2X \Rightarrow \boxed{l = 1 + X + (1+2X)^{1/2}}$

(+ve sign since $l > X$)

max H is (from the differential equation) where $x = 1 + X$.

[5]
[new but standard]

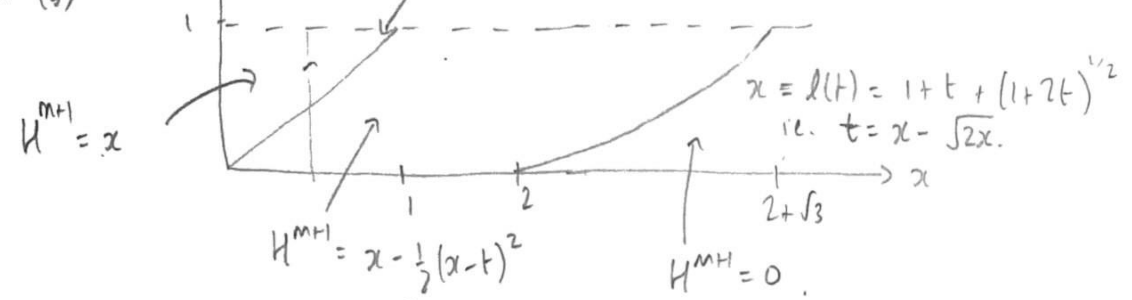
(c)(i) Basal shear stress was $\rho g \sin\theta \left(1-\cot\theta \frac{\partial s}{\partial x}\right) H$ and sliding speed was $C(\rho g \sin\theta)^m \left(1-\cot\theta \frac{\partial s}{\partial x}\right)^m H^m$

So, after non-dimensionalising, the product is $\left(1-\mu \frac{\partial s}{\partial x}\right)^{m+1} H^{m+1}$, and with $\mu = 0$ this is just H^{m+1}.

So if erosion is proportional to this product we have $\frac{\partial b}{\partial t} = -\beta H^{m+1}$ for some β. [2].

(ii) The ice thickness evolves quasi-statically, depending on X as in (b), and we can just consider $0 < t < 1$, when $X = t$, then double it to get the whole erosion (by symmetry) [2].

From (b)



So, for any given x,
$$\frac{\partial b}{\partial t} = \begin{cases} 0 & t \le x - \sqrt{2x} \\ -\beta\left[x - \frac{1}{2}(x-t)^2\right] & x - \sqrt{2x} < t \le x. \\ -\beta x & t > x. \end{cases}$$ [3]

Integrating (along vertical lines in diagram above) with $b = 0$ at $t = 0$ gives

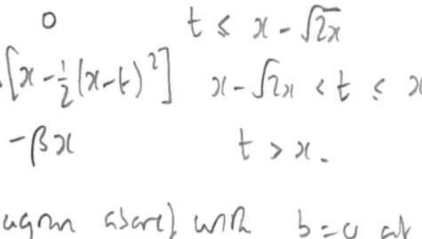
for $0 < x < 1$, $b = -\beta\left[xt + \frac{1}{6}(x-t)^3\right]_0^x - \beta\left[xt\right]_x^1 = -\beta\left[x - \frac{1}{6}x^3\right]$

for $1 < x < 2$, $b = -\beta\left[xt + \frac{1}{6}(x-t)^3\right]_0^1 = -\beta\left[x - \frac{1}{6}x^3 + \frac{1}{6}(x-1)^3\right]$

for $2 < x < 2+\sqrt{3}$, $b = -\beta\left[xt + \frac{1}{6}(x-t)^3\right]_{x-\sqrt{2x}}^1 = -\beta\left[x - x^2 + \frac{2}{3}\sqrt{2}\,x^{3/2} + \frac{1}{6}(x-1)^3\right]$

for $x > 2+\sqrt{3}$, $b = 0$. [3]

Then double for final shape.



[picture not needed – partial marks for doing some relevant integrals even if incorrect]

[new].