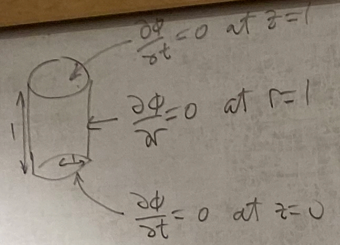
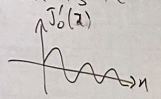


$$\phi(r,z,t) = A_{n,m} e^{-i\omega t} J_0(\xi_n r)\sin(m\pi z)$$

$n, m \in \mathbb{N}$

$$J_0'(\xi_n) = 0$$

i.e. ξ_n is the n^{th} positive root of $J_0' = -J_1$



$$f'(1) = -i\omega\varepsilon$$

$$\rightarrow A\omega J_0'(\omega) = -i\omega\varepsilon$$

$$\boxed{A = \frac{-i\varepsilon}{J_0'(\omega)}}$$

$$\boxed{\phi(r,t) = \frac{-i\varepsilon J_0(\omega r)}{J_0'(\omega)} e^{-i\omega t}}$$

If ω is (close to) a zero of J_0' we get resonance.

Natural frequencies $\omega_{n,m}^2 = m^2\pi^2 + \xi_n^2$

(b) $\frac{\partial\phi}{\partial r} = -i\omega\varepsilon e^{-i\omega t}$ at $r = 1 + \cancel{\varepsilon e^{-i\omega t}}$

$$\frac{\partial\phi}{\partial r}\left(1+\varepsilon e^{-i\omega t}, z, t\right) = \underbrace{\frac{\partial\phi}{\partial r}(1,z,t)}_{O(\varepsilon)} + \overbrace{\varepsilon e^{-i\omega t}\frac{\partial^2\phi}{\partial r^2}\left(\quad\right)}^{O(\varepsilon^2)} + \dots$$

Try $\phi(r,t) = f(r)e^{-i\omega t}$

wave eqn $\quad f'' + \frac{1}{r}f' + \omega^2 f = 0 \qquad$ let $\omega r = \xi$

$$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} + f = 0$$

bounded as $r \to 0$

$$f(r) = A J_0(\omega r) + \cancel{B Y_0(\omega r)}$$

(c) $f(r) = A J_0(\omega r) + B Y_0(\omega r)$

$$f'(1) = -i\omega\varepsilon$$

$$\phi(r,t) = e^{-i\omega t}\left(A J_0(\omega r) + B Y_0(\omega r)\right)$$

as $r\to\infty$, $\phi(r,t) \sim e^{-i\omega t}\sqrt{\frac{2}{\pi\omega r}}\left(A\cos\left(\omega r - \frac{\pi}{4}\right) + B\sin\left(\omega r - \frac{\pi}{4}\right)\right)$

$$= \sqrt{\frac{2}{\pi\omega r}}\left[\left(\frac{A-iB}{2}\right)e^{i(\omega r - \omega t - \frac{\pi}{4})} + \left(\frac{A+iB}{2}\right)e^{-i(\omega r + \omega t + \frac{\pi}{4})}\right]$$

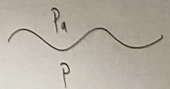
outward travelling

inward travelling

choose $A = -iB$

$p_a - p = \gamma \kappa$

$\kappa = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = \eta_{xx} + \text{h.o.t.}$



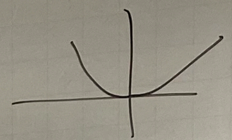
$\eta(x,t) = \frac{a}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon|k|}\, e^{ikx} \cos(\omega(k)t)\, dk$

$= \frac{a}{2\pi}\,\mathrm{Re}\int_{-\infty}^{\infty} e^{-\varepsilon|k|}\, e^{i(\omega(k)-kV)t}\, dk$

NB $f(k)$ and $\omega(k)$ are both even functions

$\omega(k) = \sqrt{\frac{\gamma}{\rho}}\,|k|^{3/2}$

$\frac{d}{dk}|k| = \mathrm{sgn}(k)$



$\therefore\ \omega'(k) = \frac{3}{2}\sqrt{\frac{\gamma}{\rho}}\,|k|^{1/2}\,\mathrm{sgn}(k)$

$\omega''(k) = \frac{3}{4}\sqrt{\frac{\gamma}{\rho}}\,\frac{1}{|k|^{1/2}}$

$\psi(k) = \omega(k) - kV$

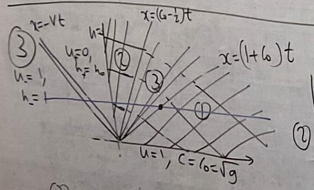
k_* satisfies $\psi'(k_*) = 0$, i.e. $\frac{3}{2}\sqrt{\frac{\gamma}{\rho}}\,|k_*|^{1/2}\,\mathrm{sgn}(k_*) = V > 0$

so $k_* > 0$; $\boxed{k_* = \frac{4}{9}\frac{\rho}{\gamma}V^2}$

Then $\psi(k_*) = \sqrt{\frac{\gamma}{\rho}}\cdot\frac{8}{27}\left(\frac{\rho}{\gamma}\right)^{3/2}V^3 - \frac{4}{9}\frac{\rho}{\gamma}V^3 = \boxed{-\frac{4}{27}\frac{\rho}{\gamma}V^3 = \psi(k_*)}$

$\psi''(k_*) = \omega''(k_*) = \frac{3}{4}\sqrt{\frac{\gamma}{\rho}}\,\frac{3}{2}\sqrt{\frac{\gamma}{\rho}}\,\frac{1}{V} = \frac{9}{8}\frac{\gamma}{\rho V} > 0$

$f(k_*) = e^{-\varepsilon|k_*|} \to 1$ as $\varepsilon \to 0$



$\boxed{① \ u=1,\ c=c_0} = \sqrt{g}$

② $u - 2c = 1 - 2c_0$

$\boxed{u=0,\ c = c_0 - \frac{1}{2}} > 0$

③ expansion fan still $\boxed{u - 2c = 1 - 2c_0}$

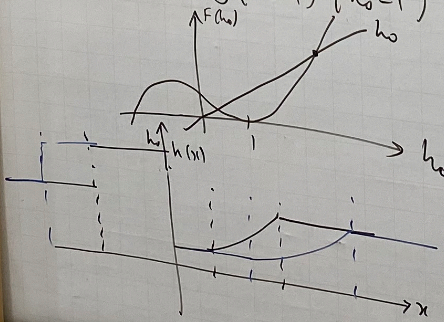
the characteristics must be straight lines thru origin

on which $\boxed{\frac{x}{t} = u + c}$ Solve simultaneously for u & c

$\boxed{u = \frac{2}{3}\left(\frac{x}{t} - c_0\right) + \frac{1}{3}, \quad c = \frac{1}{3}\left(\frac{x}{t} + 2c_0 - 1\right)}$

[NB continuous at $\frac{x}{t} = c_0 + 1,\ c_0 - \frac{1}{2}$]

(c) $2h_0 = g(h_0+1)(h_0-1)^2 = F(h_0)$



Unique root for $h_0 > 1$

$h = \frac{c^2}{g}$