

C3.11 Riemannian Geometry: Essential formulae

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Throughout we have that

- (M, g) and (N, h) are Riemannian manifolds;
- $p \in M$;
- X, Y, Z, W are tangent vectors at $p \in M$ or vector fields on M as appropriate.

Essential formulae

Global formulae

Pullback

For $f : M \rightarrow (N, h)$ smooth, pullback f^*h :

$$(f^*h)_p(X, Y) = h_{f(p)}(df_p(X), df_p(Y)).$$

Levi-Civita connection

In this section ∇ is the Levi-Civita connection on (M, g) .

$$\nabla_{aX+bY}Z = a\nabla_XZ + b\nabla_YZ, \quad \nabla_X(Y+Z) = \nabla_XY + \nabla_XZ, \quad \nabla_X(aY) = a\nabla_XY + X(a)Y$$

$$X(g(Y, Z)) = g(\nabla_XY, Z) + g(Y, \nabla_XZ), \quad \nabla_XY - \nabla_YX = [X, Y]$$

Exponential map

In this section we consider the exponential map $\exp_p : B_\epsilon(0) \subseteq T_pM \rightarrow (M, g)$.

$$d(\exp_p)_0 = \text{id}$$

Gauss Lemma:

$$g_{\exp_p(X)}(d(\exp_p)_X(X), d(\exp_p)_X(Y)) = g_p(X, Y).$$

Curves and variations

In this section:

- $\alpha : [0, L] \rightarrow (M, g)$ is a curve;
- f is a variation of α ;
- V_f is the variation field of f .

Length of α :

$$L(\alpha) = \int_0^L |\alpha'(t)| dt = \int_0^L \sqrt{g(\alpha'(t), \alpha'(t))} dt.$$

Energy of α :

$$E(\alpha) = \int_0^L |\alpha'(t)|^2 dt.$$

Length and energy satisfy

$$L(\alpha)^2 \leq LE(\alpha)$$

with equality if and only if $|\alpha'|$ is constant

Energy of f :

$$E_f(s) = \int_0^L \left| \frac{\partial f}{\partial t}(s, t) \right|^2 dt.$$

For proper variations f of α :

$$\frac{1}{2} E_f'(0) = - \int_0^L g(V_f, \nabla_{\alpha'} \alpha') dt.$$

If α is a geodesic and f is a proper variation:

$$\frac{1}{2} E_f''(0) = - \int_0^L g(V_f'' + R(V_f, \gamma') \gamma', V_f) dt = \int_0^L g(V_f', V_f') - R(V_f, \gamma', \gamma', V_f) dt.$$

Curvature

In this section:

- ∇ is the Levi-Civita connection on (M, g) ;
- $\{E_1, \dots, E_n\}$ is an orthonormal frame for $T_p M$.

Riemann curvature operator:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Riemann curvature tensor:

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

Properties of R :

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z) = R(Z, W, X, Y),$$

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0.$$

Sectional curvature of 2-plane $\sigma = \text{Span}\{X, Y\}$:

$$K(\sigma) = K(X, Y) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

Ricci curvature:

$$\text{Ric}(X, Y) = \sum_{i=1}^n R(E_i, X, Y, E_i)$$

Scalar curvature S :

$$S(p) = \sum_{i,j=1}^n R(E_i, E_j, E_j, E_i) = \sum_{j=1}^n \text{Ric}(E_j, E_j)$$

(M, g) has constant sectional curvature K if and only if

$$R(X, Y, Z, W) = K(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)).$$

Jacobi fields

In this section:

- γ is a geodesic in (M, g) with $\gamma(0) = p$;
- J is a Jacobi field along γ ;
- \exp_p is the exponential map at p .

Jacobi equation:

$$J'' + R(J, \gamma')\gamma' = 0.$$

If $J(0) = 0$ then

$$J(s) = d(\exp_p)_{s\gamma'(0)}(sJ'(0)).$$

Multiplicity of conjugate point $\gamma(T)$ to p along γ :

$$\text{mult}(\gamma(T)) = \dim \text{Ker}(d(\exp_p)_{T\gamma'(0)}).$$

Riemannian submanifolds

In this section:

- (M, g) is an n -dimensional Riemannian submanifold of (N, h) ;
- ∇^M, ∇^N are Levi-Civita connections on $(M, g), (N, h)$ respectively;
- ξ is a normal vector field along M ;
- R^M, R^N are Riemann curvature operators on $(M, g), (N, h)$ respectively.

Second fundamental form

Second fundamental form:

$$B(X, Y) = \nabla_X^N Y - \nabla_X^M Y = (\nabla_X^N Y)^\perp.$$

Properties of B :

$$B(Y, X) = B(X, Y) \quad \text{and} \quad B(aX + bY, Z) = aB(X, Z) + bB(Y, Z).$$

Shape operator:

$$g(S_\xi(X), Y) = g(B(X, Y), \xi) = g(X, S_\xi(Y)).$$

$$S_\xi(X) = -(\nabla_X^N \xi)^T.$$

Gauss equation:

$$g(R^N(X, Y)Z, W) = g(R^M(X, Y)Z, W) + g(B(X, Z), B(Y, W)) - g(B(X, W), B(Y, Z))$$

Minimality (for $\{E_1, \dots, E_n\}$ orthonormal frame on (M, g)):

$$\sum_{i=1}^n B(E_i, E_i) = 0 \quad \Leftrightarrow \quad \text{tr}(S_\xi) = 0 \text{ for all } \xi.$$

Hypersurfaces

In this section:

- (M, g) is an n -dimensional hypersurface in (N, h) ;
- ν is a normal vector field along M ;
- $\{E_1, \dots, E_n\}$ is an orthonormal basis of $T_p M$ consisting of eigenvectors of shape operator S_ν .

Principal curvatures λ_i and principal directions E_i :

$$S_\nu E_i = \lambda_i E_i.$$

Gaussian curvature:

$$K_M(p) = \prod_{i=1}^n \lambda_i(p)$$

Mean curvature:

$$H_M(p) = \frac{1}{n} \sum_{i=1}^n \lambda_i(p)$$

If $(N, h) = (\mathbb{R}^{n+1}, g_0)$, derivative of Gauss map satisfies:

$$d\nu_p = -S_{\nu(p)}$$

Local coordinate formulae

In this section:

- (x_1, \dots, x_n) are local coordinates on n -dimensional (M, g) ;
- X_1, \dots, X_n are corresponding coordinate vector fields;
- ∇ is Levi-Civita connection and R is Riemann curvature operator of (M, g) .

Metric:

$$g_{ij} = g(X_i, X_j).$$

Christoffel symbols:

$$\nabla_{X_i} X_j = \sum_{k=1}^n \Gamma_{ij}^k X_k.$$

Riemannian volume form:

$$\sqrt{\det(g_{ij})} dx_1 \wedge \dots \wedge dx_n.$$

Geodesics

Geodesic equations:

$$x_k'' + \sum_{i,j=1}^n \Gamma_{ij}^k x_i' x_j' = 0 \quad \text{for } k = 1, \dots, n.$$

Let

$$L = \frac{1}{2} \sum_{i,j} g_{ij} x_i' x_j'.$$

Then geodesic equations are equivalent to:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial x_k'} \right) - \frac{\partial L}{\partial x_k} = 0 \quad \text{for } k = 1, \dots, n$$

Symmetry lemma and proposition

In this section:

- $f : A \subseteq \mathbb{R}^2 \rightarrow (M, g)$ is a smooth function of $(u, v) \in A$ for suitable A ;
- X is a vector field along f .

Symmetry Lemma:

$$\nabla_{\frac{\partial f}{\partial u}} \frac{\partial f}{\partial v} = \nabla_{\frac{\partial f}{\partial v}} \frac{\partial f}{\partial u}.$$

Symmetry Proposition

$$\left(\nabla_{\frac{\partial f}{\partial u}} \nabla_{\frac{\partial f}{\partial v}} - \nabla_{\frac{\partial f}{\partial v}} \nabla_{\frac{\partial f}{\partial u}} \right) X = R\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)X.$$

Useful but not essential formulae

Global formulae

Koszul formula for Levi-Civita connection ∇ on (M, g) :

$$g(\nabla_X Y, Z) = \frac{1}{2} \left(X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \right).$$

If (M, g) has constant sectional curvature K then

$$\text{Ric} = (n - 1)K \quad \text{and} \quad S = n(n - 1)K.$$

Hyperbolic n -space (\mathcal{H}^n, g) :

$$\mathcal{H}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0\},$$

and g is the restriction of

$$\sum_{i=1}^n dx_i^2 - dx_{n+1}^2.$$

Poincaré disk model (B^n, g) of hyperbolic n -space:

$$B^n = \{(y_1, \dots, y_n) \in \mathbb{R}^n : \sum_{i=1}^n y_i^2 < 1\} \quad \text{and} \quad g = \sum_{i=1}^n \frac{4dy_i^2}{(1 - \sum_{i=1}^n y_i^2)^2}.$$

Upper-half space model (H^n, g) of hyperbolic n -space:

$$H^n = \{(z_1, \dots, z_n) \in \mathbb{R}^n : z_n > 0\} \quad \text{and} \quad g = \sum_{i=1}^n \frac{dz_i^2}{z_n^2}.$$

Local formulae

In this section:

- (x_1, \dots, x_n) are local coordinates on n -dimensional (M, g) ;
- X_1, \dots, X_n are corresponding coordinate vector fields;
- ∇ is Levi-Civita connection of (M, g) .

Christoffel symbols:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^n g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

For curve α in (M, g) and smooth functions a_1, \dots, a_n along α :

$$\nabla_{\alpha'} \sum_{i=1}^n a_i X_i = \sum_{k=1}^n \left(a'_k + \sum_{i,j=1}^n \Gamma_{ij}^k a_i x'_j \right) X_k.$$