B4.4 Fourier Analysis Consultation Session 4 TT23

Example 1 Suppose $u \in \mathscr{D}'(\mathbb{R})$ is 2π periodic. What is its *Fourier expansion* and in what sense does it hold? Briefly explain the connection to the Fourier transform of u.
Let $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$. Show that $\rho_\varepsilon * u$ is 2π periodic and that the n-th Fourier coefficients satisfy $c_n(\rho_\varepsilon * u) \to c_n(u)$ as $\varepsilon \searrow 0$.
Next prove that

$$c_n(\rho_\varepsilon * u) = c_n(u) \int_{-1}^{1} \rho(y)\, \mathrm{e}^{\mathrm{i}\varepsilon n y}\, \mathrm{d}y$$

and deduce that $\left|c_n(\rho_\varepsilon * u)\right| \leq \left|c_n(u)\right|$. Let $p(x) \in \mathbb{C}[x]$ be a polynomial in one indeterminate. Show that $p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) u$ is a 2π periodic tempered distribution and express its Fourier coefficients in terms of $c_n(u)$.

Example 2 Let u be a regular 2π periodic distribution on $\mathbb{R}$. Show that its n-th Fourier coefficient for $n \neq 0$ satisfies

$$c_n(u) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(u(x) - u(x + \frac{\pi}{n})\right) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x,$$

and deduce that $c_n(u) \to 0$ as $|n| \to \infty$.
Show that if v is a 2π periodic distribution of order 0, then its Fourier coefficients $c_n(v)$ are bounded:

$$\sup_{n \in \mathbb{Z}} \left| c_n(v) \right| < \infty.$$

Can this bound be strengthened to $c_n(v) \to 0$ as $|n| \to \infty$?

Example 3 Let $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R})$ (so u, $u' \in \mathsf{L}^2_{\mathrm{loc}}(\mathbb{R})$) be 2π periodic. Show that its Fourier coefficients c_n satisfy

$$\sum_{n\in\mathbb{Z}}\left(1+n^2\right)|c_n|^2 = k\int_0^{2\pi}\left(|u|^2+|u'|^2\right)\mathrm{d}x.$$

for some constant k that should be determined.

Example 4 Find $\mathcal{F}_{x\to\xi}\left(\sin x\right)$ and use it in conjunction with a differentiation rule to find $\mathcal{F}_{x\to\xi}\left(\mathrm{sinc}(x)\right)$. *HINT:* $\int_{\mathbb{R}}\mathrm{sinc}(x)\,\mathrm{d}x=\pi$. Next state a convolution rule and use it to find $\mathcal{F}_{x\to\xi}\left(\mathrm{sinc}^2(x)\right)$.

Calculate for each $x \in \mathbb{R}$ the infinite series

$$\sum_{n\in\mathbb{Z}}\mathrm{sinc}^2(x+n\pi).$$