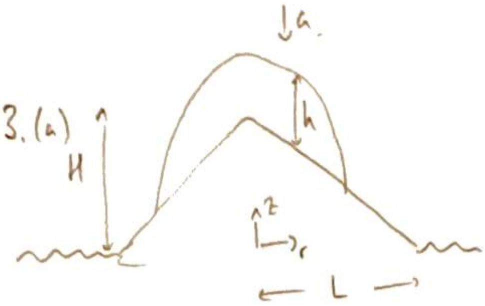
3. (a)



$$\boxed{\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rq) = a} \qquad b = H\left(1 - \frac{r}{L}\right)$$

(1) $0 = -\frac{\partial p}{\partial r} + \frac{\partial \tau}{\partial z}$ (2) $0 = -\frac{\partial p}{\partial z} - \rho g$ (3) $\frac{\partial u}{\partial z} = 2A|\tau|^{n-1}\tau$ with $p = \tau = 0$ at $z = \underbrace{b+h}_{s}$, $u = 0$ at $z = b$.

(2) $\Rightarrow p = \rho g(b+h-z)$

(1) $\Rightarrow \frac{\partial \tau}{\partial z} = \rho g\left(\frac{\partial b}{\partial r} + \frac{\partial h}{\partial r}\right) = -\rho g\left(\frac{H}{L} - \frac{\partial h}{\partial r}\right)$. so $\tau = \rho g(b+h-z)\underbrace{\left(\frac{H}{L} - \frac{\partial h}{\partial r}\right)}_{\text{assumed positive } \left(-\frac{\partial s}{\partial r}\right)}$

(3) $\Rightarrow u = 2A(\rho g)^n\left(\frac{H}{L} - \frac{\partial h}{\partial r}\right)^n\left[\frac{h^{n+1}}{n+1} - \frac{(b+h-z)^{n+1}}{n+1}\right]$

so $q = \int_b^s u\,dz = 2A(\rho g)^n\left(\frac{H}{L} - \frac{\partial h}{\partial r}\right)^n\left[\frac{h^{n+2}}{n+1} - \frac{h^{n+2}}{(n+2)(n+1)}\right] = \boxed{\frac{2A(\rho g)^n}{n+2}\underbrace{\left(\frac{H}{L} - \frac{\partial h}{\partial r}\right)^n}_{\left(-\frac{\partial s}{\partial r}\right)^n} h^{n+2}}$ [6]

Non-dimensionalising, we write $a = [a]\hat{a}$, $r = L\hat{r}$, $h = [h]\hat{h}$, $t = [t]\hat{t}$, $q = [q]\hat{q}$, and choose

$$[q] = [a]L = \frac{2A(\rho g)^n}{n+2}\frac{[h]^{2n+2}}{L^n}, \text{ and } \boxed{[t] = \frac{[h]}{[a]}}$$

$$\hookrightarrow \boxed{[h] = \left(\frac{[a]L^{n+1}(n+2)}{2A(\rho g)^n}\right)^{\frac{1}{2n+2}}}$$

Then, dropping hats, the dimensionless mass conservation eqn becomes $\boxed{\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(rh^{n+2}\left(\lambda - \frac{\partial h}{\partial r}\right)^n\right) = a}$

where $\boxed{\lambda = \frac{H}{[h]}}$ is the dimensionless height of the island.

[3]

[Slightly modified scalings].

(b)(i) If $\lambda \gg 1$, it means the island is quite tall, so the height of the ice surface is primarily controlled by the bedrock. Since we generally expect net accumulation to increase with altitude (colder temperatures $\Rightarrow$ less melting), it is reasonable to think that a should depend primarily on r in this case (as a proxy for ice surface height), and we expect it to be larger for smaller r.

If $a = \lambda(r_* - r)$, a is positive for $r < r_*$. We'd expect r_* to <u>decrease</u> as the climate <u>warms</u>.

[3]

[Marks available for other [New] physically reasonable arguments].

(ii) If $a = \lambda(r_* - r)$ and we take $\lambda \gg 1$, then the steady state approximately satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r h^{n+2}\right) = r_* - r$$

so $r h^{n+2} = \frac{1}{2} r_* r^2 - \frac{1}{3} r^3 \Rightarrow \boxed{h = \left[\frac{1}{2} r\left(r_* - \frac{2}{3} r\right)\right]^{\frac{1}{n+2}}}$ for $0 < r < \frac{3}{2} r_*$

The ice thickness is positive for $r < \frac{3}{2} r_*$, so it takes up the whole island if $\boxed{r_* > \frac{2}{3} =: r_c}$ [3]

[Similar]

(iii) If $r_* > \frac{2}{3}$ the ice flux at $r=1$ (where the ice reaches the ocean) is $q \approx \lambda h^{n+2} = \frac{\lambda}{2} r\left(r_* - \frac{2}{3} r\right)\Big|_{r=1}$

and the total iceberg flux is therefore $\underbrace{2\pi r q}_{\text{from integrating around the island}} = \boxed{\pi \lambda \left(r_* - \frac{2}{3}\right)}$.

[3]

[New]

(c)(i) If $\lambda \ll 1$, the ice surface height is primarily determined by the ice thickness, so it makes most sense to take $a(h)$, with a increasing with h. If $a = 4(h^4 - h_*^4)$, we'd expect h_* to increase as the climate warms.

[2]

(ii) If $\lambda \ll 1$, and $n=1$, the steady state approximately satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(\underbrace{-r h^3 \frac{\partial h}{\partial r}}_{\frac{1}{4}\frac{\partial}{\partial r}(h^4)}\Big) = 4\left(h^4 - h_*^4\right)$$

Let $y = h^4$ and $r = \frac{1}{4}x$, then $\frac{1}{x}\frac{\partial}{\partial x}\left(x \frac{\partial y}{\partial x}\right) + y = h_*^4$

i.e. $\frac{\partial^2 y}{\partial x^2} + \frac{1}{x}\frac{dy}{dx} + y = h_*^4$, with bounded solution $y = h_*^4 + A J_0(x)$.

We need $y = \frac{\partial y}{\partial x} = 0$ at $x = 4 r_m$ (zero ice thickness and zero ice flux)

so need $h_* + A J_0(4 r_m) = 0$ and $J_0'(4 r_m) = 0$.

Hence we need $4 r_m = \alpha$, the first positive zero of $J_0'(x)$ (we don't want the flux to be zero at any earlier x).

Hence $\boxed{r_m = \frac{\alpha}{4}}$, and $\boxed{h = h_* \left(1 - \frac{J_0(4r)}{J_0(4r_m)}\right)^{1/4}}$

[5]

[New, but with similar elements to a problem sheet qn.]