Prelims Probability

Sheet 1 — MT23

1. How many ways are there to order the letters of the word ABSTEMIOUSLY? In how many of these do the letters A and B remain next to each other? In how many do the six vowels (AEIOUY) remain in alphabetical order?

2. Celia the centipede has 100 feet, 100 socks, and 100 shoes. How many orders can she choose from to put on her socks and shoes? (She must put a sock on foot i before putting a shoe on foot i.)

3. A fair die is rolled nine times. What is the probability that 1 appears three times, 2 and 3 each appear twice, 4 and 5 once and 6 not at all?

4. Let $[n+1]$ be the set defined by $[n+1] = \{1, 2, \ldots, n+1\}$. Call a subset of $[n+1]$ with $r+1$ distinct elements an $(r+1)$-subset. How many $(r+1)$-subsets of $[n+1]$ have $(k+1)$ as their largest element? Deduce that

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

5. Starting from the axioms of probability, $\mathbf{P}_1 - \mathbf{P}_3$ from lectures[1], deduce the following results. (Feel free to make use of any set relations that you need.)
   (a) $\mathbb{P}(\emptyset) = 0$,
   (b) $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$,
   (c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ (so a generalisation of $\mathbf{P}_3$ to the case $A \cap B \neq \emptyset$).

6. Let A, B and C be events. The event "A and B occur but C does not" may be expressed as $A \cap B \cap C^c$.
   (a) Find an expression for the event "at least one of B and C occurs but A does not".
   (b) Show that the probability of the event in (a) is equal to

$$\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C).$$

   (c) How many of the numbers $1, 2, \ldots, 600$ are divisible by 5 or 7 but not by 4?

[1] see, for instance, page 6 of the lecture notes

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7. **(The birthday problem).** There are n people present in a room. Assume that people's birthdays are equally likely to be on any day of the year.
   (a) What is the probability that at least two of them celebrate their birthday on the same day? How large does n need to be for this probability to be more than $\frac{1}{2}$? (Ignore leap years.)
   (b) What is the probability that at least one of them celebrates their birthday on the same day as you? How large does n need to be for this probability to be more than $\frac{1}{2}$?

8. A confused college porter tries to hang n keys on their n hooks. He does manage to hang one key per hook, but other than this all arrangements of keys on hooks are equally likely. Let A_i be the event that key i is on the correct hook.

   We would first like to find the probability that at least one key is on the correct hook, which is $\mathbb{P}(\cup_{i=1}^n A_i)$. The generalisation of 5(c) to the case of n events is

$$\mathbb{P}\left(\bigcup_{1\le i\le n} A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{1\le i<j\le n} \mathbb{P}(A_i \cap A_j) + \sum_{1\le i<j<k\le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\mathbb{P}\left(\bigcap_{1\le i\le n} A_i\right).$$

   This is the *inclusion-exclusion formula.*
   (a) Explain why $\mathbb{P}(A_1) = \frac{(n-1)!}{n!}$ and $\mathbb{P}(A_1 \cap A_2) = \frac{(n-2)!}{n!}$.
   (b) The second sum on the right-hand side above is over all pairs (i, j) which satisfy the condition $1 \le i < j \le n$. Write down the number of such pairs.
   (c) By generalising the ideas in (a) and (b), find the probability that at least one key is on the correct hook.
   (d) Now let $p_n(r)$ denote the probability that exactly r keys are on the correct hook, for $0 \le r \le n$. Find $p_n(0)$.
   Show that
$$p_n(r) = \frac{1}{r!}\sum_{k=0}^{n-r} \frac{(-1)^k}{k!}.$$
   (e) *(Optional.)* Use induction to prove the inclusion-exclusion formula.

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