Prelims Probability

Sheet 4 — MT23

1. Let $\mathbb{P}(X = k) = 1/n$ for $k = 1, 2, \ldots, n$. Find the mean and variance of X.

2. Suppose that the discrete random variables X and Y have joint probability mass function given by

Y \ X	-1	0	1
-1	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{2}{27}$
0	$\frac{2}{27}$	$\frac{6}{27}$	$\frac{1}{27}$
1	$\frac{3}{27}$	$\frac{2}{27}$	$\frac{4}{27}$

   Find the marginal distributions of X and Y. What is the covariance of X and Y? Are X and Y independent?

3. Suppose that X and Y are independent Poisson random variables with parameters λ and μ respectively. Find
   (a) the joint probability mass function $\mathbb{P}(X = k, Y = m)$;
   (b) $\mathbb{P}(X + Y = n)$ (what is this distribution?);
   (c) $\mathbb{P}(X = k | X + Y = n)$ (what is this distribution?);
   (d) $\mathbb{E}[X | X + Y = n]$.

4. Let X and Y be independent random variables, both with Geometric(p) distribution.
   (a) Find $\mathbb{P}(X = k | X + Y = n + 1)$, for $k \in \{1, 2, \ldots, n\}$.
   (b) Find the distribution of $\min\{X, Y\}$. [*Hint: consider* $\mathbb{P}(\min\{X, Y\} > k)$, *and see Question 3(a) of sheet 3.*]

5. (a) A set of lecture notes has n pages. The number of typos on each page is a Poisson random variable with parameter λ, and is independent of the number of typos on all other pages. What is the expected number of pages with no typos?
   (b) When reading the notes, you detect each typo with probability p, independently of detecting others. Let M denote the number of tyops on a particular page and let D denote the number that you detect on that page. Write down $\mathbb{P}(D = k | M = m)$. Hence, for each $k \geq 0$, find $\mathbb{P}(D = k)$.

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6. Let X and Y be discrete random variables. Show that the following two definitions of independence of X and Y are equivalent:

   (i) For all $x, y \in \mathbb{R}$,
   $$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y);$$

   (ii) For all $A, B \subseteq \mathbb{R}$,
   $$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

   Show that if X and Y are independent, then also $f(X)$ and $g(Y)$ are independent for any functions $f, g \colon \mathbb{R} \to \mathbb{R}$.

7. Solve the following recurrence relations:

   1. $u_{n+1} = 3u_n + 2$ with $u_0 = 0$.
   2. $u_{n+1} = 2u_n + n$ with $u_0 = 1$.
   3. $u_{n+1} - 5u_n + 6u_{n-1} = 2$ with $u_0 = u_1 = 1$.
   4. $u_{n+1} - 3u_n + 2u_{n-1} = 1$ with $u_0 = u_1 = 0$.

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