Prelims Probability

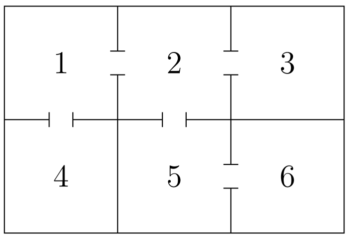
Sheet 5 — MT23

1. A bug jumps around the vertices of a triangle. At every jump, it moves from its current position to either of the other two vertices with probability 1/2 each (independently of how it arrived at its current position). The bug starts at vertex 1. Let p_n be the probabilility that it is at vertex 1 after n jumps.

   (a) Find the value of p_n for each n. [*Hint: find an appropriate first-order linear recurrence relation.*]

   (b) What happens to p_n as $n \to \infty$?

2. The diagram below shows the floor plan of a house with six rooms: in room 1 is a mouse which will change rooms every minute, first moving at $t = 1$ and choosing a door to an adjoining room at random. In room 6 is a sleeping but hungry cat which will instantly wake if the mouse should enter. How long on average can we expect the mouse to survive?



3. (**Gambler's ruin, symmetric case.**) A gambler starts a game with a bankroll of £n where $n \in \{1, 2, \ldots, M-1\}$. At each step of the game, he wins £1 with probability 1/2 and loses £1 with probability 1/2, independently for different steps. The game ends when the gambler's bankroll reaches £0 or £M.

   In lectures we saw that the probability the gambler finishes with £M is n/M.

   (a) What is the expected amount of money that the gambler has at the end of the game?

   (b) Suppose we know that the gambler ends the game with £M. What is the conditional probability that he won £1 on the first step?

   (c) Let e_n be the expected length of the game. Find e_n for each n. For which n is e_n largest?

Matthias Winkel: winkel@stats.ox.ac.uk or winkel@maths.ox.ac.uk

4. (a) Suppose that X has a geometric distribution with parameter p. Show that the probability generating function of X is

$$G_X(s) = \frac{ps}{1-(1-p)s}, \qquad \text{for } |s| < \frac{1}{1-p}.$$

(b) Use this to calculate the mean and variance of X.

5. (a) A fair coin is tossed n times. Let r_n be the probability that the sequence of tosses never has a head followed by a head. Show that

$$r_n = \frac{1}{2}r_{n-1} + \frac{1}{4}r_{n-2}, \quad n \geq 2.$$

Find r_n using the conditions $r_0 = r_1 = 1$. Check that the value you get for r_2 is correct.

(b) Let X be the number of coin tosses needed until you first get two heads in a row. (Note that $X \geq 2$.) Find the probability mass function of X.

(c) Find the probability generating function of X. Use this to calculate the mean of X. *(You may wish to check that your answer agrees with what you got for Question 6 on Problem Sheet 3!)*

(d) Let Y be the number of coin tosses needed until you first see a tail followed by a head. On any two particular coin tosses, the probability of seeing the pattern TH is $1/4$, the same as the probability of seeing the pattern HH. Therefore $\mathbb{P}(Y = 2) = \mathbb{P}(X = 2) = 1/4$. Find $\mathbb{P}(Y > n)$ for $n \geq 1$ and compare it to $\mathbb{P}(X > n)$. Is your answer surprising?

6. (*Optional. If you liked the coupon collector problem on Problem Sheet 3, you may enjoy this question too!*)
Consider a symmetric random walk on a cycle with N sites, labelled $0, 1, 2, \ldots, N-1$. A particle starts at site 0, and at each step it jumps from its current site i to one of its two neighbours $i+1 \mod N$ and $i-1 \mod N$ with equal probability (independently of how it arrived at its current position).

(a) Find the expected number of steps until every site has been visited. [*Hint: just after a new site has been visited, what does the set of visited sites look like? The value $e_1 = M - 1$ from Question 3(c) may be useful!*]

(b) For each $k = 1, \ldots, N-1$, what is the probability that k is the last site to be visited? [*Hint: before visiting site k, the walk must visit either site $k-1$ or site $k+1$. What needs to happen from that point onwards?*]

Matthias Winkel: winkel@stats.ox.ac.uk or winkel@maths.ox.ac.uk